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$H_2$ Control Performance Limitations for SIMO Feedback Control Systems

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$\mathcal{H}_2$ Control Performance Limitations for SIMO Feedback Control Systems

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This thesis was typeset by the author using MiKTeX 2.3 and WinEdt 5.3. The typeface used for the main text is Palatino.
Dedicated to:
Jamil Karim and Nasrukan,
from whom I owe mathematics.
This thesis is devoted to a research area that studies the fundamental performance limitation and trade-off of feedback control, a subject intensively developed in the linear time-invariant feedback systems, beginning with the classical work of Bode in the 1940s on logarithmic sensitivity integrals. In modern control design, the studies on performance limitation serve as an appendage tool since they help a control system designer specifies reasonable control objectives, and understands the intrinsic limits and the trade-off between conflicting design considerations.

In this thesis we quantify and characterize the fundamental performance limitations arise in $\mathcal{H}_2$ optimal tracking and regulation control problems of single-input multiple-output (SIMO) linear time-invariant (LTI) feedback control systems. In tracking problem, the control performance is measured by the tracking error response, possibly under control input constraint, with respect to a step reference input. While in regulation problem, the performance is measured by the energy of the measurement output simultaneously with that of the control input and sensitivity constraints, against an impulsive disturbance input.

Our primary interest is not on how to find the optimal or robust controller. Rather, we are interesting in relating the optimal performance with some simple characteristics of the plant to be controlled. In other words, we provide the analytical closed-form expressions of the optimal performance in terms of dynamics and structure of the plant. The analytical expressions, however, constitute guidelines for designing an easily controllable plant in practical situations, from which the control system designer may rely in determining the optimal design parameters and reasonable control strategies.

We mostly focus our attention on tracking and regulation problems of discrete-time systems. Toward the existing results of continuous-time systems, we make small corrections and perform a few extensions. We then reformulate and resolve both problems in delta domain. An analysis on the continuity property shows that we can completely recover the continuous-time expressions from the delta domain expressions stand point as sampling
time approaches zero. Frankly speaking, we provide comprehensive and unified expressions on the characterization of the control performance limitations in the $\mathcal{H}_2$ tracking and regulation problems. Furthermore, our analytical expressions show that the optimal tracking and regulation performances are explicitly characterized by the plant’s non-minimum phase zeros and unstable poles as well as the plant gain. We confirm the effectiveness of the derived expressions by several illustrative examples. We also show how to apply the analytical expressions to practical applications including the control of three-disk torsional system, the determination of the optimal parameters in inverted pendulum system, and the sensor selection in magnetic bearing system. In addition, by exploiting the delta domain expressions we derive the analytical closed-form expressions of the optimal tracking and regulation performances for delay-time systems and by invoking our discrete-time LTI results we provide the similar expressions to approximate the optimal tracking performance for sampled-data systems.
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1 Introduction

1.1 Background

In classical paradigm, the optimal control problem reflects its main concern on the controller design. A bunch of efficient algorithms and methods for finding the optimal or robust controller which stabilizes the feedback interconnection subject to some predetermined specifications is the definite result of this approach. The best controller is usually characterized by a set of Riccati equations or linear matrix inequalities. In tracking and regulation control problems of minimum phase and stable systems, it is well-known that the non-minimum phase zeros and unstable poles of the open-loop system limit the best achievable control performance. In other words, the existence of such kind of zeros and poles affects the easiness of the plant to be controlled.

In recent control applications more challenging control problems have been emerged, since they have to be solved by taking into account some physical constraints such as measurement accuracy, control input effort, sampling period, delay time, and channel capacity. Hence, it is very interesting to investigate the control performance accomplishment in feedback control system under some physical constraints. The importance of this topic lies in its ability to provide answers to a fundamental question: Is any minimum phase and stable plant always easy to control under physical constraints in practical applications?

Under this perspective, since last decade there is a growing research activity on solving the optimal control problem by directing the main objective on the plant design instead of the controller design. This new paradigm assumes that the best controller is always available for a given plant. Hence, a more desirable plant, i.e., an easily controllable plant that satisfies some specifications such as having minimal tracking error, minimal energy regulation, its poles lie in a prescribed region, etc., are requested. Obviously, instead of having Riccati equations or linear matrix inequalities, existence of the analytical closed-form expressions of the pursuing performance in terms of the
plant dynamics and structure are inevitable. The analytical closed-form expression characterized by plant dynamics and structure, however, reveals the fundamental performance limitations inherently affect the respecting control problem and subsequently constitute a class of easily controllable plants.

In modern control design, the knowledge on performance limitation constitutes an extremity, since it enables the control system designer specify the reasonable control objectives and understand the performance limitations inherently affect the design. The understanding of performance limitations thus promises to be of intrinsic as well as practical value. In practical situations, the characterization of the fundamental performance limitations aids the designer to achieve the control performance by determining the optimal design parameters, e.g., providing the optimal length of pendulum in inverted pendulum system, and in selecting the reasonable control strategies such as selecting the sensor in magnetic bearing system. Wider plant design changes can also be considered based on the characterization including changing the apparatus, relocating actuators and sensors, adding new equipment to dampen the disturbance, adding new actuators and sensors, and changing the control objectives.

1.2 Fundamental Performance Limitations

In practical control design, it is often important to know how easy the plant can be controlled, what control structure should be used, and how might the process be changed to improve control [54]. These three questions are related to the inherent control characteristics of the process itself. These characteristics comprise some intrinsic limitations of the process to be controllable, i.e., to achieve acceptable control performance in maintaining the outputs within specified bounds from their reference inputs, in spite of unknown variations, using available inputs and measurements. Therefore, the performance limitation has the same essential attributes as the controllability:

- It is independent of the controller.
- It is a fundamental property of the plant (or process) alone.
- It can only be affected by plant design changes.

Fundamental performance limitations are central for all design problems but, due to Bode, often neglected in control works. A classic example to this fact is the design of the flight controller for the X-29 experimental aircraft [1]. Much design effort was done with many methods and much cost to fulfill the design criteria: The phase margin should be greater than 45° for all flight conditions. An argument based on fundamental laws shows that this specification is unfeasible.

Control system design always involves performance considerations and physical limitations, which subsequently emerges a trade-off between the
1.2 Fundamental Performance Limitations

Fig. 1.1. Feedback control systems.

high performance objective and and the need for meeting hard design constraints. Thus, having insights into fundamental performance limitations are useful for control system, since we can quantify how various constraints may limit the level of achievable performance leading to some realistic design objectives. The characterization of the best achievable performance in terms of the dynamics and structure of the system to be controlled reveals what is and, conversely, what is not achievable prior to applying any specific technique for solving the control problem. A qualitative method, i.e., simulation approach, may not satisfactorily tackle this problem, since one can never know if the result is a fundamental property of the plant, or if it depends on some set-points.

Nowadays, there are two main research directions in the study of fundamental performance limitations. First direction lies in the extensions of the well-known Bode’s integral theorem [3] to assess design constraints and performance limitations via logarithmic-type integrals. Second direction focuses on the formulations of optimal control problems to quantify and characterize the fundamental performance limits. Roughly speaking, after 1980 there have been steady stream results in studying the fundamental limitations in feedback control system. Latest results on this studies can be seen in a special issue of the IEEE Transactions on Automatic Control in August 2003 and a book that presents a comprehensive account of the modern frequency domain/input-output results on limits to performance [53].

1.2.1 Bode Integral Relation

Consider the feedback control configuration depicted by Fig. 1.1, where $P$ is the plant and $K$ is the stabilizing controller. The signals $r$, $d$, $u$, $y$, and $e := r - y$ represent the reference input, disturbance input, control input, system output, and error, respectively. The sensitivity function $S$ and the complementary sensitivity function $T$ are defined by

$$S(s) := \frac{1}{1 + P(s)K(s)}, \quad T(s) := \frac{P(s)K(s)}{1 + P(s)K(s)}.$$  \hspace{1cm} (1.1)

It is well-known that the sensitivity function and the complementary sensitivity function tell much about feedback loop since they describe the effects of process variations. It is advantageous to have a small value of $S$ to obtain small control error for commands and disturbance and a small value of $T$ to
allow large process uncertainty. Their definitions say that $S(s) + T(s) \equiv 1$ holds for any choice of $P(s)$ and $K(s)$, which means that $S$ and $T$ cannot be made small simultaneously. The loop transfer function $PK$ is typically large for small values of $s = j\omega$ and it goes to zero as $\omega$ goes to infinity. This means that $S$ is typically small for small $s = j\omega$ and close to 1 for large $\omega$. The complementary sensitivity function is close to 1 for small $s = j\omega$ and it goes to 0 as $\omega$ goes to infinity. A basic problem is to investigate if $S$ can be made small over a large frequency range. There are unfortunately severe constraints on the sensitivity function.

The pioneering work of Bode in 1945 [3, pp. 285], which is originally related to feedback amplifier design, reveals that the sensitivity reduction cannot be accomplished once a time at all frequency range of the imaginary axis. In other words, sensitivity reduction smaller than one over a particular frequency range will contribute sensitivity expansion greater than one over some other frequency range. Technically speaking, the sensitivity function, $S$, must obey the following integral relation for a linear time-invariant stable open-loop plant

$$\int_{0}^{\infty} \log |S(j\omega)| \, d\omega = 0. \quad (1.2)$$

In 1963, Bode’s theorem has been applied to the feedback control problem for the very first time by Horowitz [33]. The result of Bode has played a fundamental role in feedback design and have received renewed interest in recent years. The utility and importance of this result has motivated and led to several extensions to Bode’s theorem. It is pointed-out by Kwakernaak and Sivan [36, pp. 440–441] that if the open-loop system is asymptotically stable, then the integral could be zero, finite, and infinite, depending on the degree difference between numerator and denominator of the open-loop transfer function. Important result has been achieved by Freudenberg and Looze [22], where the Bode’s integral has been extended to the open-loop unstable case, i.e., (1.2) has been generalized to

$$\int_{0}^{\infty} \log |S(j\omega)| \, d\omega = \pi \sum_{k=1}^{n_p} p_k, \quad (1.3)$$

where $p_k (k = 1, 2, \ldots, n_p)$ are the unstable poles of the open-loop plant. It is showed by (1.3) that the integral is proportional to the unstable open-loop plant. The discrete-time versions of the sensitivity integral were further derived by Maciejowski [38], Middleton [40], Mohtadi [45], Sung and Hara [55]. The generalized Bode’s theorems for multi-variable systems have been carried-out by Chen and Nett [16] and Chen [8]. Moreover, there are some extended versions of Bode’s theorem for different class of systems, including delay-time systems [23,27], linear time-varying systems [34], nonlinear systems [65], and linear time-periodic systems [52].

The progress of the researches has also led to the development of Bode and Poisson type integrals [7, 9, 10, 32], making available a series of integral
1.2 Fundamental Performance Limitations

constraints, in either equality or inequality, on the sensitivity and complementary sensitivity functions applicable to multi-variable systems. Other relevant extensions have been pursued in [26] and [28], toward problems pertaining to sampled-data systems and filter design.

Overall, the sensitivity and complementary sensitivity integrals characterize how certain plant properties such as non-minimum phase zeros and unstable poles in the open-loop transfer function may impose constraints upon feedback design. They also serve as a benchmark for evaluating the system's performance prior to and after controller design.

1.2.2 Optimal Control Problem

The classical optimal control methods give the optimal performance with the specified criteria and constraints if the control objectives are achievable. They do not tell what to do if the objectives cannot be accomplished and they seldom give useful insight into the mechanism that cause the limitations. It is therefore in the recent years a new paradigm of control problem based on the plant design rather than controller design has received much interest. The primary objective of this control paradigm is to relate the optimal performance with the plant properties. This technique gives insight into the factors that fundamentally limit the achievable performance of a control system, since it provides the analytical closed-form expression of the optimal performance in terms of the plant dynamics and structure.

The studies on fundamental performance limitations in optimal control problem under the new paradigm have been extensively carried-out since the beginning of this decade. Most of the studies are performed in $H_2$ optimal control setting and by using the transfer function approach, and only a few numbers that executed in $H_\infty$ criterion or by adopting the state space approach (e.g., see [43, 64]). Two among the topic studies which have earned much attentions until the current state are the $H_2$ optimal tracking and regulation performance problems.

$H_2$ Tracking Performance Problem

The tracking performance is usually quantified as the integral-square (for continuous-time case) or summation-square (for discrete-time case) of the error between the output signal of the plant and the reference input signal, where the reference input $r(t)$ is the unit step function. The best tracking ability is then measured by the minimum tracking performance achievable by all controllers that stabilize the plant:

$$J^* = \inf_{K \in \mathcal{K}} \int_0^\infty |e(t)|^2 dt,$$

where $\mathcal{K}$ is the set of all stabilizing controllers. For many years, investigation on the tracking performance of the feedback system has become an important problem. In linear time-invariant (LTI) context, the first result on the
tracking step reference input of SISO stable systems can be found in [46], which show that the non-minimum phase zeros of the plant completely characterize the optimal tracking performance. Some generalization then have been made to unstable cases applicable for SIMO system [12] and MIMO system [17], including one for discrete-time system [58]. The latter has also extended the problem by considering different type of reference input signals such as sinusoidal and ramp signals. All the results suggest that in multi-variable systems, the spatial properties such as the direction of the reference input give additional limitations on the optimal performance. While, other results have regarded the preview control as the control strategy [18] and considered the sampled-data system as the platform analysis [15]. There are also a number of results which accommodate the robustness issue in the tracking problem [6, 30, 50].

As one of the performance objectives, tracking accuracy is often mutually conflicting with other performance objectives, e.g., minimizing the energy of the control input:

$$J^* = \inf_{K \in \mathcal{K}} \int_0^\infty (|e(t)|^2 + |u(t)|^2) \, dt.$$  \hspace{1cm} (1.5)

It means that the tracking error problem has been reexamined under condition that only finite control input energy is available. Therefore, understanding a complex trade-off among those becomes fundamental from the control performance achievability point of view. The accomplishment in this extension consist of the results for SIMO systems [31] and MIMO stable systems [14].

Frankly speaking, the investigation of the $H_2$ tracking performance limitation for LTI systems is almost complete except that we do not have any results for SIMO LTI discrete-time system, since the MIMO result is valid only for right-invertible plant, where the number of input is greater or equal to that of output. New directions related to the research in this area may take form of extending the analysis beyond LTI system, i.e., we may consider nonlinear or time-varying system, and more complex control design than unity feedback or expand the known results by new problem and application area, or by developments in novel design techniques and methods.

$H_2$ Regulation Performance Problem

In optimal regulation performance problem, the control objective is to minimize the energy of the control input, or to minimize the energy of the control input simultaneously with the energy of the system output against an impulse disturbance input $d(t)$, i.e.,

$$E^* = \inf_{K \in \mathcal{K}} \int_0^\infty |u(t)|^2 \, dt.$$  \hspace{1cm} (1.6)
1.4 Contribution of the Thesis

or

\[ E^* = \inf_{K \in \mathcal{K}} \int_0^\infty (|y(t)|^2 + |u(t)|^2) \, dt. \tag{1.7} \]

We call the former the energy regulation problem and the latter the output regulation problem. Results on $\mathcal{H}_2$ energy regulation problem can be found in [31] which is conducted for unstable/non-minimum phase SISO/SIMO plants. Equivalent results in SISO systems but articulated in term of signal-to-noise ratio constrained channels are found in [4, 42]. Meanwhile, results on $\mathcal{H}_2$ output regulation problem of minimum phase SISO/MIMO systems are presented in [14]. Note that the linear time-varying feedback stabilization has been discussed in [42]. It is well-known that the regulation performance is not only imposed by non-minimum phase zeros and unstable poles of the plant, but also by the plant gain.

Summarizing the existing results, the investigation for the minimum phase plant is almost complete, while the researches for unstable and non-minimum phase plants are not complete. Especially further investigations are required for the LTI discrete-time system.

1.3 Objectives

The primary objective of this research is two-fold. Firstly, we quantify and characterize the fundamental performance limitations may arise in the $\mathcal{H}_2$ optimal tracking and regulation performance problems pertaining to SIMO LTI feedback control system. We consider SIMO system with the reasons that MIMO system is theoretically hard and SIMO system itself is practically meaningful, since it consists of systems with single actuator and more than one sensors, which commonly appear in typical situation of control problems. Theory of fundamental performance limitations for the important special case of a single-input two-output (SITO) system can be found in [25, 60]. Secondly, we provide guidelines of plant design from the view point of control as a set of easily controllable plants in practical applications. We accomplish the research objectives by deriving the analytical closed-form expressions of the best achievable performance.

1.4 Contribution of the Thesis

In this thesis we examine two kinds of optimal control problem in SIMO LTI feedback control system, namely $\mathcal{H}_2$ optimal tracking and regulation control problems. Instead of proposing a novel algorithm to obtain the optimal or robust controller, we investigate the fundamental performance limitations may arise in the control problems, where the limits are represented by analytical closed-form expressions in terms of the plant properties. Analytical expressions rather than numerical solutions are quite useful not only to understand
feedback control systems but also to characterize a set of easily controllable plants in practical situations, where the formulae can provide guidelines for plant design from the viewpoint of control.

The main contribution of this thesis dwells on the result of $H_2$ optimal regulation performance problem for SIMO discrete-time systems. Overall, we supply a comprehensive and unified expressions by deriving the analytical expressions of the optimal tracking and regulation performances for continuous-time, discrete-time, and delta domain cases. To deal with as general as possible situation, we consider a class of non-minimum phase and unstable systems.

(i) We provided the analytical closed-form expressions of the optimal performance in tracking step input signal, where the tracking ability is measured by the error between the input reference signal and the measurement output signal, possibly under control input constraint.
   - **Continuous-time Case:** The existing results of continuous-time system contain some mistakes. We correct them by explicitly accounting an additional effect caused by the plant’s unstable poles.
   - **Discrete-time Case:** We derive the analytical closed-form expressions of the optimal tracking performance for discrete-time system. The key idea proposed in this part is the application of the bilinear transformation to obtain two key lemmas from the continuous-time counterparts.
   - **Delta Domain Case:** We reformulate and resolve the tracking performance problem of discrete-time system in terms of the delta operator. The existence of the sampling time explicitly in the expression enables us to unify the all results.
   - **Delay-time Case:** We demonstrate that by exploiting the delta domain expression we can re-derive the existing result on the tracking performance of the delay-time systems.
   - **Sampled-data Case:** We provide a tiny contribution on tracking SISO sampled-data system. We derive the analytical closed-form expression of the optimal tracking performance by implementing the fast sampling technique.

(ii) We provide the analytical closed-form expressions of the optimal regulation performance against an impulsive disturbance input, where the regulation performance is measured by the energy of the measurement output, possibly under control input and sensitivity constraints.
   - **Continuous-time Case:** We complete the existing results in continuous-time case by deriving the analytical closed-form expression for non-minimum phase system and extend the problem to one with sensitivity penalty.
   - **Discrete-time Case:** We derive the analytical closed-form expressions of the optimal regulation performance for discrete-time case. How-
ever, this will be the first outcome on regulation performance limita-
tions of the discrete-time system.

- **Delta Domain Case:** We reformulate and resolve the regulation per-
formance problem of discrete-time system in terms of the delta op-
erator. We recover the continuous-time expressions by exhibiting the
continuity property.

- **Delay-time Case:** Another small contribution is that we provide the
analytical closed-form expression of the optimal energy regulation
performance for simple SIMO delay-time system, where the plant
is non-minimum phase and has only single unstable pole with pure
time delay in the input port.

Regarding the continuity property of the delta domain solution, we show
for both optimal control problems that the limiting zeros of the discretized
systems will not contribute any effects on the optimal performance provided
the sampling time is sufficiently small.

### 1.5 Chapter Organization

The remainder of this report is organized as follows. In Chapter 2 we state
some preliminaries. Here we introduce the notation used throughout this re-
port and describe the considered feedback control system configuration. De-
scription about plant decompositions and the set of all stabilizing controllers
can be found in this chapter. We put also in this chapter, the explanation
about delta operator and delta transform. Chapter 3 deals with the tracking
performance problem. We first describe the problem formulation and then
provide the analytical closed-form expressions of the optimal tracking per-
formance for continuous-time, discrete-time, delta domain, delay-time, and
sampled-data cases, respectively. Chapter 4 is devoted to the optimal regu-
lation problems, where we provide the analytical closed form expressions of
the optimal regulation performance for continuous-time, discrete-time, delta
domain, and delay-time cases, respectively. We consider some application is-
ues in Chapter 5. Here, we apply our results into three physical systems:
three-disk torsional system, inverted pendulum system, and magnetic bear-
ing system. Finally, some concluding statements are drawn in Chapter 6.
2 Preliminaries

2.1 Notation

We give a brief description of the notation used throughout this report. We denote the real set by $\mathbb{R}$ and the complex set by $\mathbb{C}$. For any $c \in \mathbb{C}$, its complex conjugate is denoted by $\bar{c}$. For any vector $u$ we shall use $u^T$, $u^H$, and $\|u\|$ as its transpose, conjugate transpose, and Euclidean norm, respectively. We call the one-dimensional subspace spanned by $u$ the direction of $u$. For any two vectors $u, v \in \mathbb{C}^n$, the angle between their directions is defined as

$$\angle(u, v) = \arccos \frac{|u^Hv|}{\|u\|\|v\|}.$$ 

For any matrix $A \in \mathbb{C}^{m \times n}$, we denote its conjugate transpose by $A^H$ and its column space by $\mathbb{R}[A]$. The cardinality of a set $S$ is denoted by $\#S$. In $s$-domain analysis, i.e., continuous-time case, let the open left half plane be denoted by $\mathbb{C}_- := \{s \in \mathbb{C} : \text{Re } s < 0\}$, the open right half plane by $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re } s > 0\}$, and the imaginary axis by $\mathbb{C}_0$. And for any matrix function $f \in \mathbb{C}^{m \times n}$ we define $f^\sim(s) := f^T(-s)$. For any signal $x(t)$, $t > 0$, we define its Laplace transform $\hat{x}(s)$ by

$$\hat{x}(s) = \mathcal{L}\{x(t)\} := \int_0^\infty x(t)e^{-st} dt.$$ 

While in $z$-domain analysis, i.e., discrete-time case, the unit circle is denoted by $\partial \mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$. We also define the following sets: $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^c := \{z \in \mathbb{C} : |z| \geq 1\}$, and $\mathbb{D}_c := \{z \in \mathbb{C} : |z| > 1\}$. Clearly, $\mathbb{D}$ and $\mathbb{D}_c$ respectively can be seen as the regions inside and outside unit circle. Furthermore, we define $f^\sim(z) := f^T(z^{-1})$. For any sequence $x(k)$, $k = 0, 1, \ldots$, we define its $\mathcal{Z}$ transform $\hat{x}(z)$ by

$$\hat{x}(z) = \mathcal{Z}\{x(k)\} := \sum_{k=0}^\infty x(k)z^{-k}.$$
Moreover, for $g(s)$ measurable in $C_0$ let define the Hilbert space

$$L_2(C_0) := \left\{ g : \|g\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \|g(j\omega)\|^2 d\omega < \infty \right\},$$

in which the inner product is defined as

$$\langle g_1, g_2 \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1^H(j\omega)g_2(j\omega) d\omega. \quad (2.1)$$

Similarly, for $f(z)$ measurable in $\partial D$ let define

$$L_2(\partial D) := \left\{ f : \|f\|_2^2 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(e^{j\theta})\|^2 d\theta < \infty \right\},$$

a Hilbert space with an inner product

$$\langle f_1, f_2 \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1^H(e^{j\theta})f_2(e^{j\theta}) d\theta. \quad (2.2)$$

Next, define for $g_1(s)$ analytic in $C_+$ and $g_2(s)$ analytic in $C_-$,

$$H_2(C_0) := \left\{ g_1 : \|g_1\|_2^2 := \sup_{\sigma > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|g_1(\sigma + j\omega)\|^2 d\omega < \infty \right\},$$

$$H_2^+(C_0) := \left\{ g_2 : \|g_2\|_2^2 := \sup_{\sigma < 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|g_2(\sigma + j\omega)\|^2 d\omega < \infty \right\},$$

and for $f_1(z)$ analytic in $\tilde{D}^c$ and $f_2(z)$ analytic in $D$,

$$H_2(\partial D) := \left\{ f_1 : \|f_1\|_2^2 := \sup_{r > 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f_1(re^{j\theta})\|^2 d\theta < \infty \right\},$$

$$H_2^+(\partial D) := \left\{ f_2 : \|f_2\|_2^2 := \sup_{r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f_2(re^{j\theta})\|^2 d\theta < \infty \right\}.$$

Then it is well-known that $H_2$ and $H_2^+$ are subspaces of $L_2$ containing functions that are analytic in $C_+$ or $\tilde{D}^c$ and $C_-$ or $D$, respectively. Also, they form an orthogonal pair of $L_2$, i.e., for any $h_1 \in H_2$ and $h_2 \in H_2^+$, $\langle h_1, h_2 \rangle = 0$.

Finally we denote by $\mathbb{R}H_{\infty}$ the class of all stable and proper rational transfer function matrices, or in discrete-time sense, the class of all rational matrix functions which are bounded and analytic in $D^c$.

### 2.2 Feedback Control Systems

In this present work, we shall consider the generic feedback configuration of finite dimensional LTI systems depicted in Fig. 2.1, which represents the
standard unity feedback and one parameter control scheme. In this setup, \( P \) denotes the SIMO LTI plant and \( K \) the stabilizing controller. The plant can be written as

\[
P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix},
\]

where \( P_i (i = 1, \ldots, m) \) are scalar transfer functions. We assume that the system is initially at rest and the feedback system is stable. We shall denote by \( P(s) \) and \( P(z) \), the transfer functions of the plant \( P \) in \( s \)-domain and \( z \)-domain, respectively. More generally, henceforth we shall use the same symbol to denote a system and its transfer function, and whenever convenient, to omit the dependence on the frequency variables \( s \) and \( z \). The signals \( r \in \mathbb{R}^m \), \( d \in \mathbb{R} \), \( u \in \mathbb{R} \), \( y \in \mathbb{R}^m \) and \( n \in \mathbb{R}^m \) are the reference input, the disturbance input, the control input, the measurement output, and the measurement noise, respectively. The signal \( e := r - y \in \mathbb{R}^m \) is the error. Hereafter, it will be assumed that all vectors and matrices involved in the sequel have compatible dimensions, and for simplicity their dimensions will be omitted.

A complex number \( z \) is said to be a zero of \( P \) if \( P(z) = 0 \). In addition, if \( z \) lies either in \( \mathbb{C}_+ \) for \( s \)-domain or \( \bar{D}_c \) for \( z \)-domain then \( z \) is said to be a non-minimum phase zero. \( P \) is said to be minimum phase if it has no non-minimum phase zero; otherwise, it is said to be non-minimum phase. On the other hand, a complex number \( p \) is said to be a pole of \( P \) if \( P(p) \) is unbounded. A pole \( p \) is said to be unstable if it lies in \( \mathbb{C}_+ \) or \( \bar{D}_c \). \( P \) is said to be stable if it has no unstable pole; otherwise, unstable. For technical reasons, it is assumed that the plant does not have zeros and poles at the same location.

### 2.3 Plant Factorization

#### 2.3.1 Coprime Factorization

Two transfer function matrices \( F, G \in \mathbb{R} \mathcal{H}_\infty \) are said to be right-coprime if they have equal number of columns and there exist matrices \( X, Y \in \mathbb{R} \mathcal{H}_\infty \) such that

\[
(X \ Y) \begin{bmatrix} F \\ G \end{bmatrix} = XF + YG = I.
\]
Or, in other words, the matrix \((F, G)^T\) is left-invertible in \(\mathbb{R}\mathcal{H}_\infty\). Analogously, two transfer function matrices \(F, G \in \mathbb{R}\mathcal{H}_\infty\) are said to be left-coprime if they have equal number of rows and there exist matrices \(X, Y \in \mathbb{R}\mathcal{H}_\infty\) such that

\[
(F \ G) \begin{pmatrix} X \\ Y \end{pmatrix} = FX + GY = I,
\]

or equivalently, the matrix \((F, G)\) is right-invertible in \(\mathbb{R}\mathcal{H}_\infty\).

A plant transfer function \(P \in \mathbb{R}\mathcal{H}_\infty\) is said to have right-coprime factorization if \(P = NM^{-1}\), where \(N\) and \(M\) are right-coprime in \(\mathbb{R}\mathcal{H}_\infty\). Similarly, \(P \in \mathbb{R}\mathcal{H}_\infty\) is said to have left-coprime factorization if \(P = \tilde{M}^{-1}\tilde{N}\), where \(\tilde{N}\) and \(\tilde{M}\) are left-coprime in \(\mathbb{R}\mathcal{H}_\infty\). Here, it is tacitly assumed that \(M\) and \(\tilde{M}\) be square and non-singular. Then, \(P \in \mathbb{R}\mathcal{H}_\infty\) is said to have doubly-coprime factorizations if

\[
P = NM^{-1} = \tilde{M}^{-1}\tilde{N},
\]

where \(N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty\), and there exist \(X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty\) that satisfy the double Bezout identity

\[
\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I,
\]

or equivalently,

\[
\begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = I.
\]

If the transfer function of \(P\) is determined by

\[
P = \begin{bmatrix} AB \\ CD \end{bmatrix},
\]

then the transfer function of the eight matrices are determined by

\[
N = \begin{bmatrix} A + BF \\ C + DF \end{bmatrix}, \quad M = \begin{bmatrix} A + BF \\ F \end{bmatrix},
\]

\[
\tilde{N} = \begin{bmatrix} A + HC \\ C \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} A + HC \\ C \end{bmatrix},
\]

\[
X = \begin{bmatrix} A + BF & -H \\ C + DF & I \end{bmatrix}, \quad Y = \begin{bmatrix} A + BF & -H \\ F & 0 \end{bmatrix},
\]

\[
\tilde{X} = \begin{bmatrix} A + HC & -(B + HD) \\ F & I \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} A + HC & -H \\ F & 0 \end{bmatrix},
\]

where matrix \(F\) is chosen such that \((A + BF)\) is stable and matrix \(H\) is selected such that \((A + HC)\) is stable.

Based on the coprime factorization approach, the set of all controllers \(K\) which stabilizes \(P\) is characterized by Youla parameterization [21, 59].
2.3 Plant Factorization

\[ \mathcal{K} := \{ K : K = (Y - MQ)(NQ - X)^{-1} \] 
\[ = (Q \tilde{N} - \tilde{X})^{-1}(\tilde{Y} - Q \tilde{M}) ; \quad Q \in \mathbb{R} \mathcal{H}_\infty \}. \] (2.8)

Note that if \( P \) is tall, i.e., \( P \) is an SIMO plant, then \( N \) and \( \tilde{N} \) are tall, \( M \) and \( \tilde{X} \) are scalar, \( \tilde{M} \) and \( X \) are square, \( Y \) and \( \tilde{Y} \) are fat. Obviously, \( K \) and \( Q \) are fat.

For our analysis, we denote
\[ N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_m \end{bmatrix} , \] (2.9)
where \( N_i (i = 1, \ldots, m) \) are scalar transfer functions. Furthermore, we may also define that \( p \) is the pole of \( P \) if and only if there is a unitary vector \( w \) such that \( \tilde{M}(p)w = 0 \). The unitary vector \( w \) is called the pole direction associated with \( p \).

2.3.2 Inner-outer Factorization

A transfer function \( N \), not necessarily square, is called an inner if \( N \) is in \( \mathbb{R} \mathcal{H}_\infty \) and \( N^* N = I \) for all \( s = j\omega \) or \( z = e^{j\theta} \) and is called co-inner if \( N \in \mathbb{R} \mathcal{H}_\infty \) and \( N N^* = I \). A transfer function \( M \) is called outer if \( M \) is in \( \mathbb{R} \mathcal{H}_\infty \) and has a right inverse which is analytic in \( C_+ \) or \( \bar{D}^c \). For an arbitrary \( P \in \mathbb{R} \mathcal{H}_\infty \),
\[ P = \Theta_i \Theta_o , \] (2.10)
where \( \Theta_i \) is inner and \( \Theta_o \) is outer, is defined as an inner-outer factorization of \( P \). We call \( \Theta_i \) the inner factor and \( \Theta_o \) the outer factor. If the transfer function of \( P \) is given by (2.7), then the transfer functions of the discrete-time inner and outer factors, respectively, are determined by [35]
\[ \Theta_i(z) = \begin{bmatrix} A - BF \\ C - DF \end{bmatrix} D^{-1}_{s=\bar{s}} , \] (2.11)
\[ \Theta_o(z) = \begin{bmatrix} A \\ B \\ D_{s} \end{bmatrix} F_{s} \] (2.12)
where \( D_{s} \) be an appropriate surjective matrix satisfying \( D_{s}^T D_{s} = D^T D + B^T \mathcal{P} B \), and \( F = (R + B^T \mathcal{P} B)^{-1}(B^T \mathcal{P} A + S^T) \), with \( \mathcal{P} \) is the solution of the discrete-time algebraic Riccati equation
\[ \mathcal{P} = A^T \mathcal{P} A - (A^T \mathcal{P} B + S)(R + B^T \mathcal{P} B)^{-1}(B^T \mathcal{P} A + S^T) + Q , \]
and \( Q = C^T C, \quad R = D^T D, \quad S = C^T D. \)
2.4 Delta Transforms

A book that provides a comprehensive account on delta operator is [44]. The delta operator $\delta$ is defined as the following forward difference

$$\delta \triangleq q^{-1}T,$$

where $q$ is the forward shift operator commonly used in discrete-time case and $T > 0$ is the sampling time. For any sequence $x(k)$, $k = 1, 2, \ldots$, delta operator gives

$$\delta x(k) = (q^{-1}T)x(k) = \frac{q x(k) - x(k)}{T} = \frac{x(k+1) - x(k)}{T}. \quad (2.13)$$

By taking the Z transform of above equation we obtain

$$\hat{\delta}x(z) = \frac{z^{-1}}{T}\hat{x}(z). \quad (2.14)$$

We may say that in $z$-plane the delta operator will translate a point $z \in \mathbb{C}$ one unit to the left and then scale it by factor of $\frac{1}{T}$. Later, the variable $\delta$ is used as the delta operator variable and is analogous to the Laplace variable $s$ for continuous-time systems and the Z transform variable $z$ for discrete-time systems. We then obtain the relationship between variable $z$ and variable $\delta$ as follows,

$$\delta = \frac{z-1}{T}, \quad (2.15)$$

$$z = T\delta + 1. \quad (2.16)$$

For any sequence $x(k)$ we define its delta transform by

$$D\{x(k)\} = \hat{x}_T(\delta) := T \sum_{k=0}^{\infty} x(k)(T\delta + 1)^{-k}, \quad (2.17)$$

or equivalently,

$$\hat{x}_T(\delta) = T\hat{x}(z)|_{z=T\delta+1}. \quad (2.18)$$

For $T > 0$ we define the following sets: $\partial \mathbb{D}_T = \{\delta \in \mathbb{C} : |T\delta + 1| = 1\}$, $\mathbb{D}_T := \{\delta \in \mathbb{C} : |T\delta + 1| < 1\}$, $\mathbb{D}_T^c := \{\delta \in \mathbb{C} : |T\delta + 1| \geq 1\}$, and $\bar{\mathbb{D}}_T := \{\delta \in \mathbb{C} : |T\delta + 1| > 1\}$. It is obvious that $\partial \mathbb{D}_T$ can be seen as a circle centered at $\delta = (-\frac{1}{T}, 0)$ with radius $\frac{1}{T}$. Respectively, $\mathbb{D}_T$ and $\mathbb{D}_T^c$ can be interpreted as regions inside and outside the circle. See Fig. 2.2.

For $h(\delta)$ measurable in $\partial \mathbb{D}_T$ a Hilbert space $\mathcal{L}_2$ is defined as

$$\mathcal{L}_2(\partial \mathbb{D}_T) := \left\{ h : \|h\|_2^2 := \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \| h \left( e^{j\omega T} - 1 \right) \|^2 \ d\omega < \infty \right\},$$
equipped with inner product
\[ \langle h_1, h_2 \rangle := \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} h_1^H \left( e^{j\omega T} - 1 \right) h_2 \left( e^{j\omega T} - 1 \right) d\omega. \] (2.19)

In a similar manner we may construct the orthogonal pairs \( \mathcal{H}_2(\partial \mathbb{D}_T) \) and \( \mathcal{H}_2^\perp(\partial \mathbb{D}_T) \). The well-known Parseval’s identity is also derived in this domain:
\[ \| x_T(k) \|_2^2 = \| \hat{x}_T(\delta) \|_2^2, \]
where \( x_T(k) := x(kT) \) and \( \| x_T(k) \|_2^2 = T \sum_{k=0}^{\infty} |x_T(k)|^2 \).

Let \( F(z) \) be given and define \( G_T(\delta) := F(T\delta + 1) \). Then by setting \( \theta = \omega T \), it is not difficult to verify that the following \( \mathcal{H}_2 \) norms relation holds:
\[ \| G_T(\delta) \|_2^2 = \frac{\| F(z) \|_2^2}{T}. \] (2.20)

This norm property plays an important role in our subsequent derivation, particularly in the derivation of the optimal tracking and regulation performances in delta domain.

Finally, for any matrix function \( f \in \mathbb{C}^{m \times n} \) we define \( f^\sim(\delta) := \frac{f^T}{T} \).
In this chapter we formulate and solve the optimal tracking performance problems for SIMO LTI continuous-time and discrete-time systems. We study first the tracking error problem and then extend the results to the tracking error problem under control input penalty, possibly for stable and unstable systems. Our primary attention in this work is on discrete-time case since the continuous-time results are already available. Existing results on tracking performance limitation of continuous-time systems are initially established in [12, 31]. We shall cite these results with corrections since in some parts they contain misleading. To solve the tracking error problem for discrete-time system we mainly encouraged by the continuous-time results of [12]. The approach used in [12] can be explained as follows. Based on Youla parameterization of all stabilizing controllers and the inner-outer factorization of the plant, an $H_2$ optimal control problem is built. By invoking a lemma, the general formula for optimal tracking error in terms of inner-outer factors is derived. And by applying another lemma, the analytical closed-form expressions of the optimal tracking performance are then obtained. Since we intend to bring this approach into discrete-time setting, the existence of two key lemmas in $z$-domain is inevitable.

Subsequently, we reformulate and resolve the tracking problem in the delta domain, from which we then show the continuity property, i.e., we show that we can completely recover the continuous-time result from the delta domain result stand point by approaching the sampling time to zero. Additionally, we also benefit our results for SIMO discrete-time systems to derive the optimal tracking performance of sampled-data systems. Applications of our results on the tracking problem of three-disk torsional system and inverted pendulum system can be found in Chapter 5. In the latter case we show that how the analytical closed-form expression of the optimal performance can be exploited in determining the optimal parameters of inverted pendulum system. We start this chapter by describing the tracking problem formulation and then we provide our results in continuous-time and discrete-time, respectively.
3.1 Tracking Performance Problem

In this section we provide the formulation of the $H_2$ optimal tracking performance problems, which consist of the tracking error problem and that under control input penalty. For both problems we consider the feedback control configuration depicted by Fig. 3.1, where we assume that the system’s output $y$ is resulted solely from the reference input $r$. Let the input and output sensitivity functions be defined by

$$S_i := (1 + KP)^{-1},$$
$$S_o := (I + PK)^{-1},$$

respectively. We consider step functions as the reference input signal, defined by

$$r(t) = \begin{cases} \nu, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad \hat{r}(s) = \frac{\nu}{s},$$
$$r(k) = \begin{cases} \nu, & k \geq 0 \\ 0, & k < 0 \end{cases}, \quad \hat{r}(z) = \frac{z\nu}{z - 1},$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_m)^T$ is a constant unitary vector and specifies the direction of the reference input.

3.1.1 Tracking Error Problem

For a given input signal $r$, we define the tracking error ability as

$$J_c := \int_0^\infty \| r(t) - y(t) \|^2 dt = \int_0^\infty \| e(t) \|^2 dt,$$
$$J_d := \sum_{k=0}^\infty \| r(k) - y(k) \|^2 = \sum_{k=0}^\infty \| e(k) \|^2,$$

for continuous-time and discrete-time cases, respectively. Since $\hat{e} = S_o \hat{r}$, where $S_o$ is the output sensitivity function defined in (3.2), it follows from the well-known Parseval identity that

$$J_c = \| S_o(s) \hat{r}(s) \|^2_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \| S_o(j\omega) \hat{r}(j\omega) \|^2 d\omega,$$
$$J_d = \| S_o(z) \hat{r}(z) \|^2_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \| S_o(e^{j\theta}) \hat{r}(e^{j\theta}) \|^2 d\theta.$$
We can see that the tracking abilities are measured by the $\mathcal{H}_2$ norms of the tracking error. The best achievable tracking performances by all stabilizing controllers in $\mathcal{K}$ are then determined by

$$J^*_c = \inf_{K \in \mathcal{K}} J_c,$$

$$J^*_d = \inf_{K \in \mathcal{K}} J_d.$$

Since $S_o = (X - NQ)\tilde{M}$, we can further deduce

$$J^*_c = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (X - NQ)\tilde{M} \frac{\nu}{s} \right\|_2^2,$$

$$J^*_d = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (X - NQ)\tilde{M} \frac{\nu}{z - 1} \right\|_2^2,$$

where $N$ and $\tilde{M}$ are the coprime factors of $P$ governed in (2.4) and $X$ is the corresponding Bezout matrix given in (2.5). Note that (3.8) holds by taking into account that $z$ is an inner function.

### 3.1.2 Tracking Error Problem under Control Input Penalty

We may extend the preceding problem to the $\mathcal{H}_2$ tracking error problem under control input penalty. This problem is more realistic than one without penalty on the control input, since the controller could not produce an output beyond the capability of the actuator. In this case we consider the following performance indexes

$$J_c := \int_0^{\infty} \left( \|e(t)\|^2 + |u_w(t)|^2 \right) dt,$$

$$J_d := \sum_{k=0}^{\infty} \left( \|e(k)\|^2 + |u_w(k)|^2 \right),$$

for continuous-time and discrete-time cases, respectively. Here $u_w$ is the weighted control input, i.e.,

$$u_w(t) = \mathcal{L}^{-1}\{W_u(s)\hat{u}(s)\},$$

$$u_w(k) = \mathcal{Z}^{-1}\{W_u(z)\hat{u}(z)\},$$

with stable and minimum phase weighting function $W_u$. Note that if $W_u = 0$ the problem then reduces to a tracking error one, which has been discussed in previous subsection. It follows from the Parseval’s identity that

$$J = \|\hat{e}\|_2^2 + \|\hat{u}_w\|_2^2$$

where $J$ stands either $J_c$ for continuous-time case or $J_d$ for discrete-time case. Further, we can write
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\[ J = \|S_o \dot{r}\|^2_2 + \|W_u KS_o \dot{r}\|^2_2. \]

Using coprime factorizations (2.4), double Bezout identity (2.5), and Youla parameterization (2.8) yields

\[ J = \big\| (X - NQ) \tilde{M} \dot{r} \big\|^2_2 + \big\| W_u (Y - MQ) \tilde{M} \dot{r} \big\|^2_2 \]

for a free parameter \( Q \in \mathbb{R} \mathcal{H}_\infty \). The optimal performance achievable by all stabilizing controllers then can be determined by

\[ J^* = \inf_{Q \in \mathbb{R} \mathcal{H}_\infty} \big\| \begin{bmatrix} W_u Y \\ X \end{bmatrix} - \begin{bmatrix} W_u M \\ N \end{bmatrix} Q \big\| \tilde{M} \dot{r} \big\|_2^2, \tag{3.11} \]

where \( J^* \) stands either \( J^*_c \) for continuous-time case or \( J^*_d \) for discrete-time case.

3.1.3 Plant Augmentation

To solve the tracking error problem under control penalty, we adopt the key idea of augmented plant initially introduced in [31]. An augmented plant \( P_a \) is defined as

\[ P_a = \begin{bmatrix} W_u \\ P \end{bmatrix}, \tag{3.12} \]

from which we then slightly modify the feedback control configuration depicted by Fig. 3.1 to one given by Fig. 3.2. We then obtain the corresponding step input signal \( r_a \) with direction \( \nu_a = (0, \nu^T)^T \) and the performance indexes

\[ J_{ca} := \int_0^\infty \| e_a(t) \|^2 \, dt, \tag{3.13} \]
\[ J_{da} := \sum_{k=0}^\infty \| e_a(k) \|^2, \tag{3.14} \]

where

\[ e_a := \begin{bmatrix} 0 \\ r \end{bmatrix} - \begin{bmatrix} u_w \\ y \end{bmatrix}. \]

One of the key points addressed by this strategy is that the performance indexes do not explicitly include the control input penalty \( u \). Furthermore, the corresponding right and left coprime factorizations of \( P_a \) are derived as

\[ P_a = N_a M_a^{-1} = \tilde{M}_a^{-1} \tilde{N}_a, \tag{3.15} \]

where

\[ N_a = \begin{bmatrix} W_u M \\ N \end{bmatrix}, \quad M_a = M, \quad \tilde{M}_a = \begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}, \quad \tilde{N}_a = \begin{bmatrix} W_u \\ N \end{bmatrix}. \]
and the corresponding double Bezout identity is written as

\[
\begin{bmatrix}
\tilde{X}_a & -\tilde{Y}_a \\
-\tilde{N}_a & \tilde{M}_a
\end{bmatrix}
\begin{bmatrix}
M_a Y_a \\
N_a X_a
\end{bmatrix} = I,
\]
(3.16)

where

\[
X_a = \begin{bmatrix}
1 & W_a Y \\
0 & X
\end{bmatrix},
Y_a = (0, Y), \quad \tilde{X}_a = \tilde{X}, \quad \tilde{Y}_a = (0, \tilde{Y}).
\]

For a free parameter \(Q_a = (Q_1, Q_2) \in \mathbb{R}H_\infty\), the optimal performance index \(J^*_a\), which is either \(J^*_{ca}\) or \(J^*_{da}\), can be expressed as

\[
J^*_a = \inf_{Q_a \in \mathbb{R}H_\infty} \| (X_a - N_a Q_a) \tilde{M}_a \hat{r}_a \|^2_2,
\]
(3.17)

and then

\[
J^*_a = \inf_{Q_2 \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} W_a Y \\ X \end{bmatrix} - \begin{bmatrix} W_a M \\ N \end{bmatrix} Q_2 \right\| \tilde{M} \hat{r} \|^2_2.
\]
(3.18)

The expression of \(J^*_a\) in (3.18) is exactly equivalent with that of \(J^*\) in (3.11) for the original plant \(P\). By taking into account that there is no penalty to the control input to be imposed, we can immediately follow the result of the tracking error problem.

### 3.2 Continuous-time Case

Results on tracking performance limitations of continuous-time systems can be found in [12, 14, 17, 31]. The results presented in this section mainly cited from those references. Especially for unstable case, we make essential correction on the results of [12, 31].

#### 3.2.1 Two Lemmas

We state two lemmas which play important roles in the derivation. For any function \(g(s)\) analytic in \(\mathbb{C}_+\), we define the following class of function

\[
F := \left\{ g : \lim_{R \to \infty} \max_{\theta \in [-\pi/2, \pi/2]} \frac{|g(Re^{i\theta})|}{R} = 0 \right\}.
\]
(3.19)
The above class consists of functions with restricted behavior at infinity. By this, we intend to deal with integration over a contour that becomes arbitrarily long [53]. Generally speaking, if \( g \) is analytic and bounded magnitude in \( \mathbb{C}_+ \), then \( g \) is of class \( \mathcal{F} \). The following two lemmas can be found in [12].

**Lemma 3.1** Let \( g(s) \in \mathcal{F} \) be an analytic function in \( \mathbb{C}_+ \). Denote that \( g(j\omega) = g_1(\omega) + jg_2(\omega) \). Suppose that \( g(s) \) is conjugate symmetric, i.e., \( g(s) = \overline{g(s)} \). Then

\[
g'(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g_1(\omega) - g_1(0)}{\omega^2} \, d\omega. \tag{3.20}
\]

**Proof.** See [53, pp. 50].

**Lemma 3.2** Let \( g(s) \) be a meromorphic function in \( \mathbb{C}_+ \) and has no zero or pole on \( j\omega \)-axis. Suppose that \( g(s) \) is conjugate symmetric and \( \log g(s) \in \mathcal{F} \). Also, suppose that \( z_i \in \mathbb{C}_+ \) \( (i = 1, \ldots, n_z) \) and \( p_k \in \mathbb{C}_+ \) \( (k = 1, \ldots, n_p) \) are, respectively, non-minimum phase zeros and unstable poles of \( g(s) \), all counting their multiplicities. Provided that \( g(0) \neq 0 \), then

\[
-\frac{1}{\pi} \int_{-\infty}^{\infty} \log \left| \frac{g(\omega)}{g(0)} \right| \frac{d\omega}{\omega^2} = \sum_{i=1}^{n_z} \frac{2}{z_i} - \sum_{k=1}^{n_p} \frac{2}{p_k} + \frac{g'(0)}{g(0)}. \tag{3.21}
\]

**Proof.** See [13].

**Remark:** A meromorphic function on an open subset of the complex plane is a function that is analytic in all except a set of isolated points, which are poles for the function.

### 3.2.2 Tracking Error Problem

Now we provide the analytical closed-form expressions of the optimal tracking error performance [12], i.e., we derive the minimal value of the tracking error performance (3.5):

\[
J^*_c = \inf_{K \in \mathcal{K}} \int_0^\infty \|e(t)\|^2 \, dt,
\]

which is further given by (3.7). First we give the result for stable case and then we extend it to unstable case. For the finiteness of \( J_c \) we made the following standard assumption.

**Assumption 3.1** \( N(0) \neq 0 \).

**Assumption 3.2** For \( r(t) \) in (3.3), \( \nu \in \mathbb{R}[N(0)] \).
In order for $J_c$ to be finite, it is obvious the output sensitivity function $S_o(s)$ must have a zero at $s = 0$ with input zero direction $\nu$, i.e. $S_o(0)\nu = 0$. Condition $N(0) \neq 0$ is then required to avoid any hidden pole-zero cancellation at $s = 0$ so that the open loop system has an integrator. Condition $\nu \in \mathbb{R}[N(0)]$ requires that the input signal must enter from direction lying in the column space of $N(0)$ and gives the condition of step reference signal $r(t)$ that a non-right invertible plant $P(s)$ may track.

We denote by $p_k \in \mathbb{C}_+ \ (k = 1, \ldots, n_p)$ the unstable poles of $P(s)$ and by $z_{ij} \in \mathbb{C}_+ \ (i = 1, \ldots, m, \ j = 1, \ldots, n_i)$ the non-minimum phase zeros of $P_i(s)$, and define the following index sets:

$$K_a := \{i : N_i(0) \neq 0\},$$
$$K_p := \{k : \tilde{M}(p_k)\nu = 0\},$$
$$K_{pi} := \{k : N_i(p_k) = 0\} \ (i = 1, \ldots, m).$$

Note that $K_p$ contains the index of unstable poles whose direction is coincident with that of step input signal $r(t)$. While, since $N = PM$ then $K_{pi}$ contains the index of unstable poles of $P(s)$ but not those of $P_i(s)$. The index set $K_{pi}$ will play a key role for correcting an error in existing result shown in [12,31].

**Stable Plants**

Since $P(s)$ is stable, then we may choose $N = \tilde{N} = P$, $\tilde{X} = M = 1$, $X = \tilde{M} = I$, and $Y = \tilde{Y} = 0$. We define the inner-outer factorization such that

$$P(s) = \Theta_i(s)\Theta_o(s). \quad (3.22)$$

We then may write (3.7) as

$$J_\ast^c = \inf_{Q \in \mathbb{R}_{H\infty}} \| (I - \Theta_i\Theta_o Q) \nu \|_2^2. \quad (3.23)$$

**Theorem 3.1 (Stable Plant [12])** Suppose that the SIMO plant $P(s)$ given in (2.3) is stable and $P_i(s)$ has non-minimum phase zeros $z_{ij} \ (i = 1, \ldots, m, \ j = 1, \ldots, n_i)$. Then, under Assumptions 3.1 and 3.2, the optimal tracking performance is given by

$$J_\ast^c = \sum_{i \in K_a} \nu_i^2 \sum_{j=1}^{n_i} 2 \Re \frac{z_{ij}}{|z_{ij}|^2} + \frac{1}{\pi} \sum_{i \in K_a} \nu_i^2 \int_0^\infty \log \left[ \frac{|P_i(0)|^2 \ |P(j\omega)|^2}{\|P(0)\|_2^2 \ |P_i(j\omega)|^2} \right] d\omega. \quad (3.24)$$

**Proof.** See [12].
Unstable Plants

We now extend the problem to an unstable plant. We define the inner-outer factorization such that

\[ N(s) = \Theta_i(s)\Theta_o(s). \]  

(3.25)

**Theorem 3.2 (Unstable Plant)** Suppose that the SIMO plant \( P(s) \) given in (2.3) has unstable poles \( p_k \) \((k = 1, \ldots, n_p)\) and \( P_i(s) \) has non-minimum phase zeros \( z_{ij} \) \((i = 1, \ldots, m, j = 1, \ldots, n_i)\). Then, under Assumptions 3.1 and 3.2, the optimal tracking performance is given by

\[ J^*_c = J^*_{cs} + J^*_{cu}, \]  

(3.26)

where

\[ J^*_{cs} = J_{cs1} + J_{cs2}, \]

\[ J^*_{cu} = J_{cu1} + J_{cu2} \]

with

\[ J_{cs1} := \sum_{i \in \mathbb{K}_s} \nu_i^2 \sum_{j=1}^{n_i} 2 \text{Re} \frac{z_{ij}}{|z_{ij}|^2}, \]

\[ J_{cs2} := \frac{1}{\pi} \sum_{i \in \mathbb{K}_s} \nu_i^2 \int_0^\infty \log \left( \frac{|P_i(0)|^2}{|P(0)|^2} \frac{|P(j\omega)|^2}{|P_l(j\omega)|^2} \right) \frac{d\omega}{\omega^2}, \]

\[ J_{cu1} := \sum_{i \in \mathbb{K}_s} \nu_i^2 \sum_{k \in \mathbb{K}_p} 2 \text{Re} \frac{p_k}{|p_k|^2}, \]

\[ J_{cu2} := \sum_{k, \ell \in \mathbb{K}_p} \frac{4 \text{Re} p_k \text{Re} p_\ell}{(p_k + p_\ell)|p_k p_\ell|^2} \sigma_k \sigma_\ell \left( 1 - \Theta_i^\sim(p_k)\Theta_i(0) \right) \left( 1 - \Theta_i^\sim(p_\ell)\Theta_i(0) \right), \]

and

\[ \sigma_k := \begin{cases} 1 & ; \#\mathbb{K}_p = 1 \\ \prod_{\ell \in \mathbb{K}_p, \ell \neq k} \frac{p_\ell - p_k}{p_\ell + p_k} & ; \#\mathbb{K}_p \geq 2. \end{cases} \]

**Proof.** The proof completely follows that of [12, Theorem 3.3] except for few points. Let \( \Theta_i \) in (3.25) is represented by \( \Theta_i(s) = [w_1(s), \ldots, w_m(s)]^T \). The essential correction should be made in the evaluation of the set of non-minimum phase zeros of \( w_i(s) \) which includes the unstable poles of \( P_l(s) \) but not those of \( P_i(s) \) as well as non-minimum phase zeros of \( P_l(s) \), i.e., it is immediate from Lemma 3.2 that

\[ \frac{w_i'(0)}{w_i(0)} = -\sum_{j=1}^{n_i} \frac{2 \text{Re} z_{ij}}{|z_{ij}|^2} - \sum_{k \in \mathbb{K}_p} \frac{2 \text{Re} p_k}{|p_k|^2} + \frac{1}{\pi} \int_0^\infty \log \left| \frac{w_i(j\omega)}{w_i(0)} \right| \frac{d\omega}{\omega^2}. \]
The last point was not properly recognized in the proof of [12, Theorem 3.3], and hence the term $J_{cu1}$ caused by unstable poles is missing in the theorem. Once the correction above is made, the final formula can be derived in the same way as in [12]. ■

**Remark:** We make a couple of remarks on Theorem 3.2.

- In [12, Theorem 3.3], the first three terms in Theorem 3.2 are denoted by a single quantity $J_*$, which is the optimal tracking error corresponding to the stable part of the plant $P_s$. One may presume that the closed-form expression for $J_*$ is given previously by [12, Theorem 3.2], which is devoted to the stable case. However, the inner-outer factorization in stable case is given by $P = \Theta_i \Theta_o$ and that in unstable case by $P_s = \Theta_i \Theta_o$. It indicates that the inner factors in the two cases produce different sets of non-minimum phase zeros. Theorem 3.2 provides more accurate expressions by counting separately the effects caused by non-minimum phase zeros of the plant and those by other factors, i.e., some unstable poles of the plant.

- The expression in the theorem is not complete for SIMO unstable plant since it includes an inner factor $\Theta_i(s)$ in $J_{cu2}$. We can obtain the closed-form expression of $\Theta_i(s)$ only for the SISO case and some simple/special cases. See Corollary 3.1, and for discrete-time case see Corollary 3.4.

- There exists a special case where we can see the term $J_{cu2}$ caused by unstable poles is zero even if the plant is unstable. See Corollary 3.3 for the discrete-time case.

- We can also show that $J_{cu} = J_{cu1} + J_{cu2}$ is zero when the sets of all unstable poles of $P_i(s)$ ($i = 1, \ldots, m$) are completely the same, since we can see $\mathbb{K}_p$ is empty for all $i$. Moreover, $\mathbb{K}_p$ is also empty for such a case. The case often happens for practical applications where we have only one actuator but we may add one or more extra sensors. The extra sensor can dramatically improve the tracking performance for unstable and non-minimum phase plants as seen in an example of an inverted pendulum in Section 5.2.

- As noted above, the set $\mathbb{K}_p$ contains the indices of unstable poles in which satisfy $\tilde{M}(p_k) = 0$. In some cases of SIMO system, it is not so easy to obtain $\tilde{M}$ matrix such that we can obtain the pole direction associated with $p_k$. In SISO case, without loss of generality, it can be appointed such that

$$\tilde{M}(s) = M(s) := \prod_{k=1}^{n_p} \frac{s - p_k}{s + \bar{p}_k}.$$

It is obvious that $\tilde{M}(p_k) = 0$. Also note that if $P(s)$ is an SISO plant, then $J_{cu2} = 0$ and $J_{cu1} = 0$.

In general, it is difficult to find the analytical expression of $\Theta_i$ for SIMO plant. But, in SISO case which plant has non-minimum phase zeros $z_i$ ($i =
1, \ldots, n_z), the inner factor \( \Theta_i \) in (3.25), without loss of generality, can be fixed as

\[ \Theta_i(s) = \prod_{i=1}^{n_z} \frac{z_i - s}{z_i + s}, \]

from which we get \( \Theta_i(0) = 1 \). Let define

\[ \varphi(s) := \prod_{i=1}^{n_z} \frac{z_i + s}{z_i - s}, \]

then we state the tracking performance limitations for scalar systems in the following result.

**Corollary 3.1** Let \( P(s) \) be an SISO plant which has non-minimum phase zeros \( z_i \ (i = 1, \ldots, n_z) \) and unstable poles \( p_k \ (k = 1, \ldots, n_p) \). Then,

\[ J^*_c = \sum_{i=1}^{n_z} \frac{2 \text{Re} z_i}{|z_i|^2} + \sum_{k, \ell=1}^{n_p} \frac{4 \text{Re} p_k \text{Re} p\ell (1 - \varphi(\bar{p}_k)) (1 - \varphi(p\ell))}{(p_k + p\ell) \bar{p}_k p\ell \sigma_k \sigma\ell}. \]  

(3.27)

**Example 3.1** We pick a simple example to illustrate the preceding result. We consider a scalar system whose transfer function is given by

\[ P(s) = \frac{s - z}{s - p}. \]

Firstly we fix \( p = -1 \), i.e., we consider a stable plant, and compute the optimal performance for \( z \) from 1 to 2. Fig. 3.3 confirms that non-minimum phase zero closed to imaginary axis deteriorates the performance. Secondly, we fix \( z = 1 \) and vary \( p \) from 0.1 to 2. Fig. 3.4 shows that whenever \( p \) closes to 1 then the optimal performance becomes worse since it happens almost unstable pole-zero cancelation.

For both cases, we compute the optimal performance in two manners: by using MATLAB toolbox and by using analytical closed-form expression in Corollary 3.1. We see that these two computations match well. In general, the optimal performance of a system with one unstable pole \( p \) and one non-minimum phase zero \( z \) is governed by

\[ J^*_c = \frac{2}{z} + \frac{8p}{(z - p)^2}. \]

### 3.2.3 Tracking Error Problem under Control Input Penalty

In this part we consider the tracking error problem under control penalty, which has been studied in [31]. We note that the result presented in [31] also contains a small mistake and we correct it here. We provide the minimal value of the performance index (3.9), i.e.,

\[ J^*_e = \inf_{K \in \mathcal{K}} \int_0^\infty (\|e(t)\|^2 + \|u_w(t)\|^2) \, dt, \]
Fig. 3.3. The tracking error performance for stable continuous-time system (Example 3.1).

Fig. 3.4. The tracking error performance for unstable continuous-time system (Example 3.1).

which is also given by (3.11). By adopting the augmented plant strategy we may easily solve the problem since there is no penalty to the control input to be imposed. Hence, we can immediately invoke the result of the tracking error problem, i.e., Theorem 3.2.

We make an additional assumption for system plant as follow.

Assumption 3.3  $P(s)$ has a pole at $s = 0$.

It is obvious that in order to make the steady state zero, the open-loop transfer function $P(s)K(s)$ must contain an integrator. Consequently, plant $P(s)$ must have an integrator instead of compensator $K(s)$ may have an integrator to maintain a finite control energy cost. Assumption 3.3 is then necessary.

Note that, by plant augmentation strategy, the optimal performance (3.11) can be further expressed as
Now we provide the analytical closed-form solution for unstable case. We define an inner-outer factorization such that

\[ Na(s) := \begin{bmatrix} W_a(s)M(s) \\ N(s) \end{bmatrix} = \Theta_i(s)\Theta_o(s). \] 

(3.29)

**Theorem 3.3** Suppose that the SIMO plant \( P(s) \) given in (2.3) has unstable poles \( p_k \) \( (k = 1, \ldots, n_p) \) and \( P_i(s) \) has non-minimum phase zeros \( z_{ij} \) \( (i = 1, \ldots, m, j = 1, \ldots, n_i) \). Then, under Assumptions 3.1–3.3, the optimal tracking performance under control input penalty is given by

\[
J^*_c = J^*_c s + J^*_c u,
\]

(3.30)

where

\[
J^*_c s = J^*_c s_1 + J^*_c s_2,
J^*_c u = J^*_c u_1 + J^*_c u_2
\]

with

\[
J^*_c s_1 := \sum_{i \in K_p} \nu_i^2 \sum_{j=1}^{n_i} \frac{2 \text{Re} z_{ij}}{|z_{ij}|^2},
J^*_c s_2 := \frac{1}{\pi} \sum_{i \in K_p} \nu_i^2 \int_0^\infty \log \left( \frac{|P_i(0)|^2 \|P(j\omega)|^2 + |W_a(j\omega)|^2}{\|P(i\omega)|^2} \right) \frac{d\omega}{\omega^2},
J^*_c u_1 := \sum_{i \in K_p} \nu_i^2 \sum_{k \in K_p} \frac{2 \text{Re} p_k}{|p_k|^2},
J^*_c u_2 := \sum_{k,\ell \in K_p} \frac{4 \text{Re} p_k \text{Re} p_\ell}{(p_k + p_\ell)p_k p_\ell \sigma_k \sigma_\ell} (1 - \Theta_i^\sim(p_k)\Theta_i(0))(1 - \Theta_i^\sim(p_\ell)\Theta_i(0)),
\]

and

\[
\sigma_k := \begin{cases} 1 & ; \# K_p = 1 \\ \prod_{\ell \in K_p, \ell \neq k} \frac{p_\ell - p_k}{p_\ell + p_k} & ; \# K_p \geq 2. \end{cases}
\]

**Proof.** We correct a small mistake made in [31]. We apply Theorem 3.2 to \( P_a(s) \) instead of \( P(s) \). Since the first element of \( \nu_a \) is equal to zero, then there exists no extra term in \( J^*_c \) related to \( W_a(s) \). By Assumption 3.6 we conclude that \( |P_i(0)| \) and \( \|P(0)\| \) are infinite but \( |W_a(0)| \) is finite. Then the following

\[
\frac{|P_i(0)|^2}{\|P(0)\|^2} = \frac{|P_1(0)|^2}{\|P(0)\|^2 + |W_a(0)|^2} = \frac{|P_i(0)|^2}{\|P(0)\|^2}
\]

holds. Also note that
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\[
\frac{\|P_a(j\omega)\|^2}{\|P_i(j\omega)\|^2} = \frac{\|P(j\omega)\|^2 + |W_u(j\omega)|^2}{\|P(j\omega)\|^2}.
\]

The proofs for \(J_{cs1}, J_{cu1}, \) and \(J_{cu2}\) are similar as those of Theorem 3.2.

Problem of minimizing the tracking error under control input penalty generally provides additional limits imposed by the weighting function \(W_u\), which appears in the logarithmic term in \(J_{cu2}\) and in the inner factor \(\Theta_i\) in \(J_{cu2}\). In general, we do not know the closed-form expression for \(\Theta_i\) even though the plant \(P\) is scalar. Note that if we set \(W_u(s) = 0\), i.e., non-penalty case, then we can completely recover Theorem 3.2.

The expression in Theorem 3.3 is complete for SIMO marginally stable plants in a sense that the best achievable tracking performance with control input penalty is characterized by non-minimum phase zeros and gain of the plant without using any inner-outer factorization or solving any Riccati equation. See Corollary 3.2 below for the SISO case.

For scalar system, Theorem 3.3 can be further simplified as shown in the following corollary.

Corollary 3.2 Suppose that the SISO plant \(P(s)\) is marginally stable and has non-minimum phase zeros \(z_i\) \((i = 1, \ldots, n_z)\). Then, under Assumptions 3.1–3.3, the optimal tracking performance under control penalty is given by

\[
J^*_c = \sum_{i=1}^{n_z} \frac{2 \text{Re} z_i}{|z_i|^2} + \frac{1}{\pi} \int_0^\infty \log \left[ \frac{1 + |W_u(j\omega)|^2}{|P(j\omega)|^2} \right] \frac{d\omega}{\omega^2}.
\]  

(3.31)

For some simple cases, we may still obtain the analytical closed-form expression of \(J^*_c\) even the plant \(P(s)\) is unstable as shown in the following example.

Example 3.2 We consider an SISO plant

\[
P(s) = \frac{s - 1}{s(s - p)}.
\]

The plant has one non-minimum phase zero at \(z = 1\) and possibly one unstable pole at \(z = p\), provided \(p > 0\). We calculate the tracking performance under control input penalty with \(W_u(s) = 1\). In the inner-outer factorization (3.29), fortunately we can obtain the closed-form expression of \(\Theta_i\) as follows

\[
\Theta_i(s) = \frac{1}{s^2 + \sqrt{3} + ps + 1} \left[ \frac{s(s - p)}{s - 1} \right].
\]

Hence, the optimal tracking performance is then given by

\[
J^*_c = 2 + \frac{1}{\pi} \int_0^\infty \log \left[ \frac{1}{|P(j\omega)|^2} \right] \frac{d\omega}{\omega^2} + \frac{2}{p} \left[ 1 - \frac{p + 1}{p^2 - p \sqrt{3} + p^2 + 1} \right]^2.
\]
Fig. 3.5 plots the optimal performance versus the location of unstable pole $p$. It is confirmed that whenever $p$ approaches 1 then the performance blows up. It is also interesting to note that the optimal tracking error performance with control input penalty in general is much greater than that without penalty, see Fig. 3.4 for comparison.

3.3 Discrete-time Case

In this section we provide the discrete-time solutions of the optimal tracking performance for discrete-time systems. The derivations performed in this case are parallel with that of continuous-time case.

3.3.1 Two Lemmas

The following two lemmas which are counterparts with Lemmas 3.1 and 3.2 play inevitable roles in the derivation of the optimal performance for discrete-time system. Recall the class of function $F$ defined in (3.19).

**Lemma 3.3** Let $f(z) \in F$ be an analytic function in $\mathbb{D}^c$. Denote that $f(e^{i\theta}) = f_1(\theta) + jf_2(\theta)$. Suppose that $f(z)$ is conjugate symmetric, i.e., $f(z) = \overline{f(\bar{z})}$. Then

$$f'(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1(\theta) - f_1(0)}{1 - \cos \theta} d\theta. \quad (3.32)$$

**Proof.** Consider Lemma 3.1. The key idea of this proof emerges from a fact that for every stable rational transfer function $g(s)$, analytic in $\mathbb{C}_+$, then the function
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\[ f(z) := g \left( \frac{z - 1}{z + 1} \right) \quad (3.33) \]

is analytic in \( \mathbb{D}^c \). In other words, we consider the well-known bilinear or Tustin transformation

\[ s := \frac{z - 1}{z + 1}. \quad (3.34) \]

On the boundary this transformation is

\[ j\omega := \frac{e^{j\theta} - 1}{e^{j\theta} + 1}, \]

from which we get the correspondence between \( f(e^{j\theta}) = f_1(\theta) + jf_2(\theta) \) and \( g(j\omega) = g_1(\omega) + jg_2(\omega) \). We then may express \( \omega, \omega^2, \) and \( d\omega \) as functions of \( \theta \), respectively, as follow

\[ \omega = \frac{\sin \theta}{1 + \cos \theta}, \quad \omega^2 = 1 - \cos \theta, \quad d\omega = \frac{1}{1 + \cos \theta} d\theta. \]

From (3.33), we obtain \( g'(0) = 2f'(1) \). The proof is then completed by substituting all appropriate terms to (3.20).

Lemma 3.4 Let \( f(z) \) be a meromorphic function in \( \mathbb{D}^c \) and has no zero or pole on \( \partial \mathbb{D} \). Suppose that \( f(z) \) is conjugate symmetric and \( \log f(z) \in \mathbb{F} \). Also, suppose that \( \eta_i \in \mathbb{D}^c \) \( (i = 1, \ldots, n) \) and \( \lambda_k \in \mathbb{D}^c \) \( (k = 1, \ldots, n) \) are, respectively, the non-minimum phase zeros and unstable poles of \( f(z) \), all counting their multiplicities. Provided that \( f(1) \neq 0 \), then

\[ \frac{1}{2\pi} \int_{-\pi}^\pi \log \left| \frac{f(e^{j\theta})}{f(1)} \right| \frac{d\theta}{1 - \cos \theta} = \sum_{i=1}^n \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} - \sum_{k=1}^n \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} + f'(1). \quad (3.35) \]

Proof. The proof of this lemma is similar to that of Lemma 3.3.

Remark: The bilinear transformation, or Tustin transformation, is proved very useful in relating continuous-time variable \( s \) and discrete-time variable \( z \), especially in producing the discrete-time counterpart of an continuous-time lemma. Here we present two other lemmas obtained by invoking bilinear transformation to their continuous-time counterparts: Lemmas 4.3 and 4.4, respectively.

Lemma 3.5 Let \( f(z) \in \mathbb{F} \) be an analytic function in \( \mathbb{D}^c \). Suppose that \( f(z) \) is conjugate symmetric and denote \( f(e^{j\theta}) = f_1(\theta) + jf_2(\theta) \). Then

\[ -2f'(-1) = \frac{1}{\pi} \int_{-\pi}^\pi \frac{f_1(\theta) - f_1(\pi)}{1 + \cos \theta} \, d\theta. \quad (3.36) \]

Lemma 3.6 Consider a conjugate symmetric function \( f(z) \in \mathbb{F} \). Suppose that \( f(z) \) is analytic and has no zero in \( \mathbb{D}^c \) except possibly at \( z = \infty \) with multiplicity \( v \). Provided that \( f(-1) \neq 0 \), then

\[ \frac{-2f'(-1)}{f(-1)} + 2v = \frac{1}{\pi} \int_{-\pi}^\pi \log \left| \frac{f(e^{j\theta})}{f(-1)} \right| \frac{d\theta}{1 + \cos \theta}. \quad (3.37) \]
3.3.2 Tracking Error Problem

The tracking error problem in discrete-time setting can be stated as minimizing the performance index (3.6), i.e., we determine the analytical closed-form expression of

$$J^*_d = \inf_{K \in K} \sum_{k=0}^{\infty} \|e(k)\|^2,$$

which can further be expressed as (3.8). For the finiteness of $J_d$, we make the following standard assumptions.

**Assumption 3.4** $N(1) \neq 0$.

**Assumption 3.5** For $r(k)$ in (3.4), $\nu \in \mathbb{R}[N(1)]$.

We can see that these two assumptions are the counterparts of Assumptions 3.1 and 3.2, respectively. Assumption 3.4 requires the open loop system has an integrator and Assumption 3.5 gives a condition that a non-right invertible plant $P(s)$ may track the step input signal $r(k)$.

We denote by $\lambda_k (k = 1, \ldots, n)$ the unstable poles of $P(z)$ and by $\eta_{ij} (i = 1, \ldots, m, j = 1, \ldots, n_i)$ the non-minimum phase zeros of $P_i(z)$. To facilitate our analysis we define the following index sets:

$$J_z := \{i : N_i(1) \neq 0\}$$
$$J_p := \{k : \tilde{M}(\lambda_k)\nu = 0\}$$
$$J_{pi} := \{k : N_i(\lambda_k) = 0\} (i = 1, \ldots, m).$$

**Stable Plants**

For a given stable plant $P(z)$, let introduce the inner-outer factorization of the plant $P(z)$ as follows,

$$P(z) = \Theta_i(z)\Theta_o(z).$$

Then we may write (3.8) as

$$J_d^* = \inf_{Q \in \mathbb{R}H_{\infty}} \left\| (I - \Theta_i\Theta_o Q) \frac{\nu}{z-1} \right\|^2_2. \quad (3.39)$$

**Theorem 3.4 (Stable Plant)** Suppose that the SIMO plant $P(z)$ given in (2.3) is stable and $P_i(z)$ has non-minimum phase zeros $\eta_{ij} (i = 1, \ldots, m, j = 1, \ldots, n_i)$. Then, under Assumptions 3.4 and 3.5, the optimal tracking performance is given by

$$J^*_d = \sum_{i \in J_z} \nu_i^2 \sum_{j=1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2} + \frac{1}{2\pi} \sum_{i \in J_z} \nu_i^2 \int_0^\pi \log \left[ \frac{|P_i(1)|^2 \|P(e^{i\theta})\|^2}{\|P(1)^2 \|P_i(e^{i\theta})\|^2} \right] \frac{d\omega}{1 - \cos \theta}. \quad (3.40)$$
Proof. The proof of this theorem is almost parallel with that of Theorem 3.1, which is given in [12]. In (3.39), matrix $Q$ is to be selected such that $[I - \Theta_i(1)\Theta_o(1)Q(1)]\nu = 0$ in order for $J_d^*$ to be finite. Define

$$
\Psi(z) := \begin{bmatrix} \Theta_i^-(z) \\ I - \Theta_i(z)\Theta_i^-(z) \end{bmatrix}.
$$

(3.41)

We can easily verify that $\Psi$ is a norm preserving function, i.e., $\Psi(e^{j\theta})\Psi(e^{-j\theta}) = I$. So that by pre-multiplying $\Psi$ to (3.39) we have

$$
J_d^* = \inf_{Q \in \mathbb{R}H_\infty} \left\| \frac{\Psi(I - \Theta_i\Theta_o)\nu}{z - 1} \right\|^2_2
= \inf_{Q \in \mathbb{R}H_\infty} \left\| \frac{(\Theta_i^- - \Theta_o)\nu}{z - 1} \right\|^2_2 + \left\| \frac{(I - \Theta_i\Theta_i^-)\nu}{z - 1} \right\|^2_2 \\
= \inf_{Q \in \mathbb{R}H_\infty} \left\| \frac{(\Theta_i^- - \Theta_i^-(1)) + (\Theta_i^-(1) - \Theta_o)\nu}{z - 1} \right\|^2_2 + \left\| \frac{(I - \Theta_i\Theta_i^-)\nu}{z - 1} \right\|^2_2.
$$

By noting that

$$
(\Theta_i^- - \Theta_i^-(1))\frac{\nu}{z - 1} \in \mathcal{H}_2^+,
$$

we may select $Q \in \mathbb{R}H_\infty$ such that $\Theta_i^-(1) - \Theta_o(1)Q(1) = 0$, and therefore

$$
(\Theta_i^- - \Theta_o)\frac{\nu}{z - 1} \in \mathcal{H}_2.
$$

As a result

$$
J_d^* = \inf_{Q \in \mathbb{R}H_\infty} \left\| \frac{(\Theta_i^-(1) - \Theta_o)\nu}{z - 1} \right\|^2_2 + \left\| \frac{(I - \Theta_i\Theta_i^-)\nu}{z - 1} \right\|^2_2.
$$

Because $\Theta_o$ is outer, we can always select a $Q$ such that

$$
\inf_{Q \in \mathbb{R}H_\infty} \left\| \frac{(\Theta_i^-(1) - \Theta_o)\nu}{z - 1} \right\|^2_2 = 0.
$$

And then by using $\mathcal{H}_2$ norm definition (2.2) we obtain

$$
J_d^* = \left\| \frac{(\Theta_i^- - \Theta_i^-(1))\nu}{z - 1} \right\|^2_2 + \left\| \frac{(I - \Theta_i\Theta_i^-)\nu}{z - 1} \right\|^2_2
= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} \left\{ \nu^H \Theta_i(z)\Theta_i^-(1)\nu \right\} - 1 \frac{1 - \cos \theta}{1 - \cos \theta} d\theta.
$$

Let us define $f(z) := \nu^H \Theta_i(z)\Theta_i^-(1)\nu$. Under Assumption 3.5, we obtain $f(1) = 1$. Applying Lemma 3.3 then yields

$$
J_d^* = -f'(1) = -\nu^H \Theta_i'(1)\Theta_i^-(1)\nu.
$$

(3.42)
Denote the inner factor $\Theta_i(z)$ as follows,

$$\Theta_i(z) = \left( w_1(z), w_2(z), \ldots, w_m(z) \right)^T.$$ 

According to Assumption 3.5, we may select $\nu = \Theta_i(1)$ without loss of generality. Then (3.42) becomes

$$J_d^* = - \sum_{i=1}^m w_i(1) w_i'(1) = - \sum_{i \in J_z} \nu_i^2 \frac{w_i'(1)}{w_i(1)}.$$ 

Note that $i \in J_z$ assures that $w_i(1) \neq 0$. Since $w_i(z)$ is element of inner factor $\Theta_i(z)$, then it has the same set of non-minimum phase zeros as $P_i(z)$. Hence, by invoking Lemma 3.3 we have

$$\frac{w_i'(1)}{w_i(1)} = - \sum_{j=1}^m \frac{\eta_{ij}}{|\eta_{ij} - 1|^2} + \frac{1}{2\pi} \int_{-\pi}^\pi \log \left| \frac{w_i(e^{j\theta})}{w_i(1)} \right| \frac{d\theta}{1 - \cos \theta}. \quad (3.43)$$

And by noting that $|w_i(e^{j\theta})| = |P_i(e^{j\theta})|/||P(e^{j\theta})||$ we obtain

$$\log \left| \frac{w_i(e^{j\theta})}{w_i(1)} \right| = - \frac{1}{2} \log \left[ \frac{|P_i(1)|^2 \|P(e^{j\theta})\|^2}{\|P(1)\|^2 |P_i(e^{j\theta})|^2} \right].$$

This completes the proof. ■

Theorem 3.4 demonstrates that the tracking performance for stable plant with respect to step input signal generally does not only depend on the plant non-minimum phase zeros but also on the plant gain. The first term of (3.4) denotes the effects contributed by the non-minimum phase zeros and the second term, i.e., integral term, denotes those of the plant direction. Note that the expression shares some similarities with that of continuous-time case in Theorem 3.1. Beyond already known in SISO/MIMO cases [58], the non-minimum phase zeros of SIMO plant may contribute their effects in an unusual fact depending on the input direction $\nu_i$. To explain this, let consider a simple stable single-input two-output (SITO) plant as follows,

$$P(z) = \begin{bmatrix} P_1(z) \\ P_2(z) \end{bmatrix} = \frac{1}{z - \frac{\eta}{2}} \begin{bmatrix} z - \eta \\ z \end{bmatrix}, \quad |\eta| > 1.$$ 

Assumption 3.5 requires that the step input direction $\nu$ must be in the column space of $P(1)$, i.e.,

$$\nu = \frac{1}{\sqrt{(1 - \eta)^2 + 1}} \begin{bmatrix} 1 - \eta \\ 1 \end{bmatrix},$$

which is a unitary vector. Then the first term of (3.40), denote by $J_{d\eta}$, can be written as

$$J_{d\eta}^* = \frac{(1 - \eta)^2}{(1 - \eta)^2 + (\eta - 1)^2} \frac{\eta^2 - 1}{(1 - \eta)^2 + 1} = \frac{\eta^2 - 1}{(1 - \eta)^2 + 1}.$$
which led to
\[
\lim_{\eta \to 1} J^*_d \eta = 0.
\]
This simple case thus reveals a fact different from that already known in SISO and MIMO systems, i.e, in SIMO system a non-minimum phase zero close to the point \( z = 1 \) may not give an effect.

An interesting fact may also be drawn concerning with the zero at infinity. In discrete-time system, one zero of this type corresponds to one sample delay, in which instantaneously gives one penalty, since
\[
\frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2} \to 1,
\]
as \( \eta_{ij} \to \infty \).

The interpretation of integral term is rather similar to that of Theorem 3.1, in which it concerns with the effect caused by changes in plant direction. Specifically, due to the weighting factor \( 1 - \cos \theta \) in the denominator, it is reinforced that zeros close to \( z = 1 \) may have more negative effect. This extra integral term, however, points to the main difference between MIMO and SIMO systems.

Theorem 3.4 can also be applied to recover the counterpart result for right-invertible MIMO systems provided in [58, Th. 3.1]. Suppose that the right invertible plant \( P(z) \) has non-minimum phase zeros \( \eta_i \) \( (i = 1, \ldots, n) \) and its inner factor may be formed as
\[
\Theta_i(z) = \prod_{i=1}^{n} L_i(z),
\]
where
\[
L_i(z) = \frac{z - \eta_i}{1 - \bar{\eta}_i z} u_i u_i^H + U_i U_i^H
\]
with \( u_i \) are unitary vectors iteratively computed from the zero input direction vector of \( P(z) \) one at a time, and \( U_i \) are matrices whose columns, together with \( u_i \), form an orthonormal basis of the corresponding Euclidean space, i.e., \( u_i u_i^H + U_i U_i^H = I \). We can easily verify that
\[
\Theta_i(1) = 1,
\]
\[
\Theta'_i(1) = \sum_{i=1}^{n} \frac{1 - |\eta_i|^2}{|\eta_i - 1|^2} u_i u_i^H
\]
Hence, by invoking (3.42), we get
\[
J^*_d = -\sum_{i=1}^{n} \frac{1 - |\eta_i|^2}{|\eta_i - 1|^2} \nu u_i u_i^H \nu^H = \sum_{i=1}^{n} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} \cos^2 \angle(u_i^H, \nu).
\]
Unstable Plants

For unstable plant $P(z)$ whose coprime factorizations are given by (2.4), let perform an inner-outer factorization such that

$$N(z) = \Theta_i(z)\Theta_o(z).$$  \hfill (3.44)

From Bezout identity (2.5) we obtain $\tilde{M}X - \tilde{N}Y = I$ and then $X\tilde{M} = I + NM^{-1}Y\tilde{M}$. And from its equivalence (2.6) we have $Y\tilde{M} - M\tilde{Y} = 0$ and then $M^{-1}Y\tilde{M} = \tilde{Y}$. Consequently, (3.8) becomes

$$J^*_d = \inf_{Q \in \mathcal{H}_\infty} \left\| \frac{I + \Theta_i\Theta_o(\tilde{Y} - Q\tilde{M})}{z - 1} \right\|_2.$$ \hfill (3.45)

We provide the optimal tracking performance for discrete-time unstable plants in the following theorem.

**Theorem 3.5** Suppose that the SIMO plant $P(z)$ given in (2.3) has unstable poles $\lambda_k (k = 1, \ldots, n\lambda)$ and $P_i(z)$ has non-minimum phase $\eta_{ij}$ ($i = 1, \ldots, m$, $j = 1, \ldots, n_i$). Then, under Assumptions 3.4 and 3.5, the optimal tracking performance is given by

$$J^*_d = J^*_{ds} + J^*_{du},$$ \hfill (3.46)

where

$$J^*_{ds} = J_{ds1} + J_{ds2},$$
$$J^*_{du} = J_{du1} + J_{du2}$$

with

$$J_{ds1} := \sum_{i \in J_s} \nu_i^2 \sum_{j = 1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2},$$
$$J_{ds2} := \frac{1}{2\pi} \sum_{i \in J_s} \nu_i^2 \int_0^{2\pi} \log \left[ \frac{|P_i(1)|^2}{||P_i(1)||^2} \frac{||P(e^{j\theta})||^2}{||P(e^{j\theta})||^2} \right] \frac{d\theta}{1 - \cos \theta},$$
$$J_{du1} := \sum_{i \in J_s} \nu_i^2 \sum_{k \in J_p} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2},$$
$$J_{du2} := \sum_{k, \ell \in J_p} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)(1 - \Theta_i(\bar{\lambda}_k)\Theta_i(1))(1 - \Theta_i(\bar{\lambda}_\ell)\Theta_i(1))}{h_k h_\ell (\lambda_k - 1)(\lambda_\ell - 1)(\lambda_k\lambda_\ell - 1)},$$

and

$$h_k := \begin{cases} 1 & \#_{J_p} = 1 \\ \prod_{\ell \in J_p, \ell \neq k} \frac{\lambda_k - \lambda_\ell}{1 - \lambda_\ell\lambda_k} & \#_{J_p} \geq 2. \end{cases}$$
3.3 Discrete-time Case

Proof. See Appendix A.1.

In this theorem we call \( J_{du1} \) and \( J_{du2} \) the deterioration contributed by the plant unstable poles. We remark that \( J_{du1} = 0 \) if \( P(z) \) is either a scalar plant or a SIMO plant with all unstable poles of \( P_i(z) \) (\( i = 1, \ldots, m \)) are completely the same, since we can see \( J_{p1} \) is empty for all \( i \). Furthermore, condition \( M(\lambda_k)\nu = 0 \) indicates that an unstable pole will not affect through \( J_{du1} \) unless its direction coincides with that of the input signal, i.e., \( R[N(1)] \).

Particularly, \( \lambda_k (k \notin J_p) \) does not mean that \( \lambda_k \) has no effect at all since there is still a possibility that \( k \in J_{p1} \), i.e., \( \lambda_k \) gives its contribution through \( J_{du1} \) with the similar manner the non-minimum phase zeros do. This fact, however, has improved our understanding on the roles of unstable poles in performance limitations. Beyond this, we can also make similar remarks as we have below Theorem 3.2.

We now consider two specific cases for illustrating the implication of Theorem 3.5. The proofs of the following corollaries are straightforward, thus omitted.

**Corollary 3.3** Suppose the plant \( P(z) \) satisfies \( P(1) = [P_1(1), 0, \ldots, 0]^T \), and let the input signal \( r \) be given by (3.4) with \( \nu = [1, 0, \ldots, 0]^T \). Suppose that \( P_1(z) \) is stable and has non-minimum phase zeros at \( \eta_{1j} (j = 1, \ldots, n_1) \) and \( P(z) \) has unstable poles \( \lambda_k (k = 1, \ldots, n_{\lambda}) \). Then

\[
J^*_d = \sum_{j=1}^{n_1} \left| \frac{\eta_{1j}}{\eta_{1j} - 1} \right|^2 + \sum_{k \in J_{p1}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} + \frac{1}{2\pi} \int_0^{\pi} \log \left[ \frac{\|P(e^{j\theta})\|^2}{|P_1(e^{j\theta})|^2} \right] \frac{d\theta}{1 - \cos \theta}.
\]

(3.47)

**Remark:** Note that in the above corollary we do not restrict \( P_i(z) \) (\( i \geq 2 \)) to be stable. The unstable poles of \( P(z) \) contributed by \( P_i(z) \) (\( i \geq 2 \)), if any, will not give effects through \( J_{du2} \), i.e., \( J_{du2} = 0 \), since the pole directions do not coincide with that of the input signal, that is \( M(\lambda_k)\nu \neq 0 \).

In SISO case which plant has non-minimum phase zeros \( \eta_i (i = 1, \ldots, n_{\eta}) \), the inner factor in (3.44), without loss of generality, can be fixed as

\[
\Theta_i(z) = \prod_{i=1}^{n_{\eta}} \frac{z - \eta_i}{1 - \bar{\eta}_iz},
\]

from which we get \( \Theta_i(1) = 1 \). Let define \( \phi(z) := \Theta^*_i(z)\Theta_i(1) \), i.e.,

\[
\phi(z) := \prod_{i=1}^{n_{\eta}} \frac{1 - \eta_i z}{z - \bar{\eta}_i},
\]

then we state the tracking performance limitations for scalar systems in the following corollary.
Corollary 3.4 Let $P(z)$ be an SISO plant which has non-minimum phase zeros $\eta_i$ ($i = 1, \ldots, n_\eta$) and unstable poles $\lambda_k$ ($k = 1, \ldots, n_\lambda$). Then,

$$
J^*_d = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} + \sum_{k,\ell=1}^{n_\lambda} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)(1 - \phi(\lambda_k))(1 - \phi(\lambda_\ell))}{h_k h_\ell (\lambda_k - 1)(\lambda_\ell - 1)(\lambda_k \lambda_\ell - 1)}.
$$

(3.48)

Example 3.3 Consider the single-input two-output plant $P(z)$ given by

$$
P(z) = \begin{bmatrix} P_1(z) \\ P_2(z) \end{bmatrix} = \begin{bmatrix} \frac{z-\eta}{z(1-2z)} \\ k \frac{z-1}{z(z-\lambda)} \end{bmatrix},$$

with $\eta \neq 1$, $\lambda \neq 1$, $k \geq 0$. Note that $P(1) = [\eta - 1, 0]^T$ and $P_1(z)$ is stable, i.e., the plant is in the case of Corollary 3.3. $P(z)$ has possibly one non-minimum phase zero at $z = \eta$ and one unstable pole at $z = \lambda$ with pole direction vector $w = [0, 1]^T$. We vary $\eta$ from $-2$ to $3$ and take $\nu = [1, 0]^T$. Then we calculate the optimal tracking performance $J^*_d$ by using Corollary 3.3 for $\lambda = 3$. Note that $J^*_{d(\lambda)}$ is always zero regardless the value of $\lambda$ since the pole direction does not coincide with that of input signal, i.e., $w \notin \mathbb{R}[P(1)]$. The calculations of $J^*_d$ are plotted in Fig. 3.6, which confirms that $J^*_d$ increases with $k$ and $J^*_d$ is unbounded as $\eta = 1$.

Example 3.4 We consider an SISO system represented by

$$
P(z) = \frac{z-\eta}{z(z-\lambda)},$$

Note that $P$ has relative degree $1$. First we fix $\lambda = \frac{1}{2}$, i.e., we consider a stable plant, and compute the optimal performance for $\eta$ from $-2$ to $2$. Fig. 3.7 confirms that in SISO case non-minimum phase zero closed to unit circle deteriorates the performance. Second we fix $\eta = 2$ and vary $\lambda$ from $0$ to $4$. Fig. 3.8 shows that whenever $\lambda$
closes to 2 then the optimal performance blows up since it happens almost unstable pole-zero cancellation. For both cases, we compute the optimal performance in two manners: by using MATLAB toolbox and by using analytical closed-form expression in Corollary 3.4. We see that these two computations match well. In general, the optimal performance of a discrete-time system with one unstable pole $\lambda$ and two non-minimum phase zero $\eta$ and infinity is governed by

$$J^*_d = 1 + \frac{\eta + 1}{\eta - 1} + \frac{\eta^2(\lambda^2 - 1)(\lambda + 1)^2}{(\lambda - \eta)^2}.$$ 

### 3.3.3 Tracking Error Problem under Control Input Penalty

Now we extend the problem by minimizing the tracking error simultaneously with the energy of the control input. We consider the control objective
(3.10), where its optimal value is given by (3.11). We implement the idea of
plant augmentation which is quite helpful in solving the problem.

In addition to Assumptions 3.4 and 3.5, we impose the following assump-
tion.

**Assumption 3.6** $P(z)$ has a pole at $z = 1$.

This assumption requires that the plant $P(s)$ should have an integrator to
assure the finiteness of the control energy cost. Note that the optimal perfor-
mance (3.17) can be further expressed as

$$J_d^* = \inf_{Q_a \in \mathbb{R}^{H_\infty}} \left\| [I + N_a(\bar{Y}_a - Q_a \bar{M}_a)] \frac{\nu_a}{z - 1} \right\|_2^2,$$

(3.49)

We state our result by introducing an inner-outer factorization such that

$$N_a(z) := \begin{bmatrix} W_u(z) M(z) \\ N(z) \end{bmatrix} = \Theta_i(z) \Theta_o(z).$$

(3.50)

**Theorem 3.6** Suppose that the SIMO plant $P(z)$ given in (2.3) has unstable poles
$\lambda_k (k = 1, \ldots, n_\lambda)$ and $P_i(z)$ has non-minimum phase zeros $\eta_{ij}$ ($i = 1, \ldots, m, j = 1, \ldots, n_i$). Then, under Assumptions 3.4–3.6, the optimal tracking performance un-
der control input penalty is given by

$$J_d^* = J_{ds}^* + J_{du}^*,$$

(3.51)

where

$$J_{ds}^* = J_{ds1} + J_{ds2},$$

$$J_{du}^* = J_{du1} + J_{du2}$$

with

$$J_{ds1} := \sum_{i \in J_a} \nu_i^2 \sum_{j=1}^{n_1} \left| \eta_{ij} \right|^2 - 1 \left| \eta_{ij} - 1 \right|^2,$$

$$J_{ds2} := \frac{1}{2\pi} \sum_{i \in J_a} \nu_i^2 \int_0^\pi \log \left[ \frac{|P_i(1)|^2 \|P(e^{j\theta})\|^2 + |W_u(e^{j\theta})|^2}{\|P(1)\|^2} \right] \frac{d\theta}{1 - \cos \theta},$$

$$J_{du1} := \sum_{i \in J_a} \nu_i^2 \sum_{k \in J_p} \left| \lambda_k \right|^2 - 1 \left| \lambda_k - 1 \right|^2,$$

$$J_{du2} := \sum_{k, \ell \in J_p} \frac{(\left| \lambda_k \right|^2 - 1)(\left| \lambda_\ell \right|^2 - 1)(1 - \Theta_i^*(\lambda_k) \Theta_i(1))(1 - \Theta_i^*(\lambda_\ell) \Theta_i(1))}{h_k h_\ell (\lambda_k - 1)(\lambda_\ell - 1)(\lambda_k \lambda_\ell - 1)},$$

and

$$h_k := \begin{cases} 1 & : \# J_p = 1 \\ \prod_{\ell \in J_p, \ell \neq k} \frac{\lambda_k - \lambda_\ell}{1 - \lambda_\ell \lambda_k} & : \# J_p \geq 2. \end{cases}$$
3.4 Delta Domain Case

Proof. We apply Theorem 3.5 to \( P_a(z) \) instead of \( P(z) \). Since the first element of \( \nu_a \) is equal to zero, then there exists no extra term in \( J^*_d \) related to \( W_u(z) \). By Assumption 3.6 we conclude that \(|P_i(1)|\) and \( \|P(1)\| \) are infinite but \(|W_u(1)|\) is finite. Then the following

\[
\frac{|P_i(1)|^2}{\|P_a(1)\|^2} = \frac{|P_i(1)|^2}{\|P(1)\|^2 + |W_u(1)|^2} = \frac{|P_i(1)|^2}{\|P(1)\|^2}
\]

holds. Also note that

\[
\frac{\|P_a(e^{j\theta})\|^2}{|P_i(e^{j\theta})|^2} = \frac{\|P(e^{j\theta})\|^2 + |W_u(e^{j\theta})|^2}{|P(e^{j\theta})|^2}.
\]

The proofs for \( J_{ds1}, J_{du1}, \) and \( J_{du2} \) are similar as those of Theorem 3.5. \( \blacksquare \)

Corollary 3.5 Suppose that the SISO plant \( P(z) \) is marginally stable and has non-minimum phase zeros \( \eta_i \) \((i = 1, \ldots, n_\eta)\). Then, under Assumptions 3.4–3.6, the optimal tracking performance under control input penalty is given by

\[
J^*_d = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} + \frac{1}{2\pi} \int_{0}^{\pi} \log \left[ 1 + \frac{|W_u(e^{j\theta})|^2}{|P(e^{j\theta})|^2} \right] \frac{d\theta}{1 - \cos \theta}. \tag{3.52}
\]

Corollary 3.5 reveals that whenever the SISO plant is marginally stable we have to pay extra cost, which is represented by integral term, to compensate the deterioration in performance caused by the control input constraint. This fact, however, confirms that taking the control input penalty into account generally worsens the performance.

Example 3.5 In this example we consider the following SISO marginally stable discrete-time plant

\[
P(z) = \frac{z - \eta}{(z - 1)(z^2 + z)}.\]

This plant has non-minimum phase zero at \( z = \eta \) provided \( \eta \in \bar{D}^\delta \) and relative degree 2. Fig. 3.9 shows the optimal performance \( J^*_d \), which is calculated by Corollary 3.5 and by MATLAB toolbox, with respect to the location of zero \( \eta \). If \( \eta \) tends to 1 then it happens an almost hidden pole-zero cancellation. Consequently, the optimal tracking performance grows up as warned by Assumption 3.6.

3.4 Delta Domain Case

Our motivation revisiting the tracking (and regulation) problems in \( \delta \)-domain is that the relationship between continuous-time and discrete-time results is not quite clear. For example, recall the analytical closed-form expression of the optimal tracking error for SISO stable continuous-time plant \( P(s) \) which has non-minimum phase zeros \( z_i \) \((i = 1, \ldots, n_z)\) in Corollary 3.1:
Fig. 3.9. The tracking error performance with control input penalty of a marginally stable discrete-time system (Example 3.5).

\[ J^*_c = \sum_{i=1}^{n} \frac{2 \text{Re} z_i}{|z_i|^2}, \]

and compare it with its discrete-time counterpart plant \( P(z) \) which has non-minimum phase zeros \( \eta_i \) \((i = 1, \ldots, n_\eta)\) in Corollary 3.4:

\[ J^*_d = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2}. \]

It is not easy to understand how the non-minimum phase zeros of continuous-time and discrete-time plants give their contributions in completely different ways. We implement the so-called delta operator to investigate this phenomenon. In the last few years, it has been extensively demonstrated that the delta operator is superior to the shift operator in unifying the continuous-time and discrete-time expressions. Recall Section 2.4 for short introduction on delta operator.

### 3.4.1 Two Lemmas

We begin this part by reformulating the two key lemmas, i.e., Lemmas 3.3 and 3.4, in the delta domain. Recall the set \( \mathbb{D} \) defined in (3.19).

**Lemma 3.7** Let \( h \in \mathbb{D} \) be an analytic function in \( \mathbb{D} \). Denote that \( h(e^{\omega T} - 1) = h_1(\omega) + j h_2(\omega) \). Suppose that \( h \) is conjugate symmetric, i.e. \( h(\delta) = \overline{h(\delta)} \). Then

\[ h'(0) = \frac{T^2}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{h_1(\omega) - h_1(0)}{1 - \cos \omega T} \, d\omega. \]  

(3.53)
3.4 Delta Domain Case

Proof. Consider the relationship \( h(\delta) = f(T\delta + 1) \), where \( f \) is defined in Lemma 3.3. Since \( f \) is analytic in \( \mathbb{D}^\nu \), then \( h \) is analytic in \( \mathbb{D}_T \). We can easily verify that \( h'(0) = Tf'(1) \).

Lemma 3.8 Let \( h \) be a meromorphic function in \( \bar{\mathbb{D}}_T \) and has no zero or pole on \( \partial \mathbb{D}_T \). Suppose that \( h \) is conjugate symmetric and \( \log h \in \mathbb{F} \). Also, suppose that \( \zeta_i \in \mathbb{D}_T \) \((i = 1, \ldots, n_\zeta)\), and \( \rho_k \in \mathbb{D}_T \) \((k = 1, \ldots, n_\rho)\) are, respectively, non-minimum phase zeros and unstable poles of \( h \), all counting their multiplicities. Provided that \( h(0) \neq 0 \), then

\[
\frac{T^2}{2\pi} \int_{-\pi/T}^{\pi/T} \log \left| \frac{h(e^{j\omega T-1})}{h(0)} \right| \frac{d\omega}{1 - \cos \omega T} = \sum_{i=1}^{n_\zeta} \left( \frac{2\Re \zeta_i}{|\zeta_i|^2 + T} \right) - \sum_{k=1}^{n_\rho} \left( \frac{2\Re \rho_k}{|\rho_k|^2 + T} \right) + \frac{h'(0)}{h(0)}. \tag{3.54}
\]

Proof. Use relationships \( \eta_i = T\zeta_i + 1 \) and \( \lambda_k = T\rho_k + 1 \) from (2.16).

3.4.2 Tracking Error Problem

In this part we reformulate and solve the tracking error problem in term of delta operator. We consider the following tracking measure

\[
J_\delta := T \sum_{k=0}^{\infty} ||e(k)||^2, \tag{3.55}
\]

which can be further expressed as

\[
J_\delta = \|S_\alpha(\delta) \hat{r}_T(\delta)\|^2_2.
\]

As the reference input we consider the step function (3.4) whose delta transform is given by

\[
\hat{r}_T(\delta) = T\delta + 1 \nu. \tag{3.56}
\]

We include the sampling time \( T \) in (3.55) under the following explanation. Consider the operation of a zero-order hold embedded in sampled-data system which maps a discrete sequence \( e(k) \) into continuous-time signal \( e_h(t) \). The process can be written as

\[
e_h(t) \triangleq e(k), \quad kT \leq t < (k + 1)T,
\]

from which we may define the following measure:

\[
J_h := \int_0^\infty ||e_h(t)||^2 dt.
\]
Since in this case we do vary the sampling time $T$ instead of fix it then the tracking measure of the corresponding continuous-time signal

$$J_c := \int_0^\infty \|e(t)\|^2 \, dt$$

can be fully recovered by evaluating $J_h$ as $T \to 0$. We can easily verify that

$$J_h = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \|e_h(t)\|^2 \, dt = T \sum_{k=0}^{\infty} \|e(k)\|^2 = TJ_\delta,$$

from which we deduce (3.55). For the finiteness of $J_\delta$ we impose the following standard assumptions on $P(\delta)$ and $\nu$.

**Assumption 3.7** $N(0) \neq 0$.

**Assumption 3.8** For $r(k)$ in (3.4), $\nu \in \mathbb{R}\{N(0)\}$.

We denote by $\rho_k (k = 1, \ldots, n_p)$ the unstable poles of $P(\delta)$ and by $\zeta_{ij} (i = 1, \ldots, m, j = 1, \ldots, n_i)$ the non-minimum phase zeros of $P_i(\delta)$ and introduce the following index sets:

$$I_u := \{i : N_i(0) \neq 0\}$$
$$I_p := \{k : M(\rho_k)\nu = 0\}$$
$$I_p^i := \{k : N_i(\rho_k) = 0\} (i = 1, \ldots, m).$$

We state our result for unstable plant. We define an inner-outer factorization such that

$$N(\delta) = \Theta_i(\delta)\Theta_o(\delta),$$

where we denote $\Theta^o_i(\delta) := \Theta^T_i\left(\frac{\delta}{\delta^T+1}\right)$. Note that the optimal tracking performance is determined by

$$J^*_\delta = \inf_{Q \in \mathcal{RH}_\infty} \|I + \Theta_i\Theta_o(\hat{Y} - Q\hat{M})\|_\delta^2.$$  (3.58)

**Theorem 3.7** Suppose that the SIMO plant $P(\delta)$ given in (2.3) has unstable poles $\rho_k (k = 1, \ldots, n_p)$ and $P_i(\delta)$ has non-minimum phase $\zeta_{ij} (i = 1, \ldots, m, j = 1, \ldots, n_i)$. Then, under Assumptions 3.7 and 3.8, the optimal tracking performance is given by

$$J^*_\delta = J^*_{\delta s} + J^*_{\delta u},$$

where

$$J^*_{\delta s} = J_{\delta s1} + J_{\delta s2},$$
$$J^*_{\delta u} = J_{\delta u1} + J_{\delta u2}$$

with
\[
J_{\delta 1} := \sum_{i \in \mathbb{I}_p} \nu_i^2 \sum_{j=1}^{n_i} \left( \frac{2 \Re \zeta_{ij}}{\zeta_{ij}^2} + T \right),
\]
\[
J_{\delta 2} := \frac{T^2}{2\pi} \sum_{i \in \mathbb{I}_p} \nu_i^2 \int_0^{\pi/T} \log \left[ \frac{|P_i(0)|^2}{|P_i(0)|^2} \frac{\|P(\frac{e^{i\omega T} - 1}{T})\|^2}{\|P(0)\|^2} \right] \frac{d\omega}{1 - \cos \omega T},
\]
\[
J_{\delta 1} := \sum_{i \in \mathbb{I}_p} \nu_i^2 \sum_{k \in \mathbb{I}_p} \left( \frac{2 \Re \rho_k}{|\rho_k|^2} + T \right),
\]
\[
J_{\delta 2} := \sum_{k \in \mathbb{I}_p} (T|\rho_k|^2 + 2 \Re \rho_k)(T|\rho_\ell|^2 + 2 \Re \rho_\ell) \times q_k q_\ell |\rho_k| |\rho_\ell| (1 - \Theta_\ell^{-}\Theta_i(0))(1 - \Theta_i^{-}\Theta_\ell(0)),
\]
and
\[
q_k := \begin{cases} 
1 & \#\mathbb{I}_p = 1 \\
\frac{\rho_\ell - \rho_k}{T \rho_\ell \rho_k + \rho_\ell + \rho_k} & \#\mathbb{I}_p \geq 2.
\end{cases}
\]

**Proof.** In the light of the proof of Theorem 3.5 we can immediately write \( J^*_i = J^*_1 + J^*_2 \), where
\[
J^*_1 = \left\| (\Theta_i - \Theta_i^{-}(1)) \frac{\nu}{\delta} \right\|_2^2 + \left\| (I - \Theta_\ell \Theta_i^{-}) \frac{\nu}{\delta} \right\|_2^2,
\]
\[
J^*_2 = \inf_{Q \in \mathcal{H}_2} \left\| \Theta_i^{-}(1) + \Theta_\ell (\tilde{Y} - Q \tilde{M}) \frac{\nu}{\delta} \right\|_2^2.
\]

Directly calculating the \( \mathcal{H}_2 \) norm in \( \delta \)-domain we obtain
\[
J^*_1 = -\frac{T^2}{2\pi} \int_{\pi/T}^{\pi/T} \frac{\Re \left\{ \mu H \Theta_i \left( \frac{e^{i\omega - 1}}{T} \right) \Theta_i^H(0) \nu \right\} - 1}{1 - \cos \omega T} d\omega.
\]

Application of Lemma 3.7 provides \( J^*_i = J_{\delta 1} + J_{\delta 2} + J_{\delta a 1} \). The closed-form expression of \( J^*_2 \) can be obtained by performing standard partial fraction expansion. In this process we define
\[
q(\delta) := \prod_{k \in \mathbb{I}_p} \frac{(T \delta + 1) - \lambda_k}{1 - \lambda_k (T \delta + 1)},
\]
\[
q_\rho(\delta) := \frac{(T \delta + 1) - \lambda_k}{1 - \lambda_k (T \delta + 1)},
\]
\[
q_k := \prod_{\ell \in \mathbb{I}_p, \ell \neq k} \frac{\lambda_k - \lambda_\ell}{1 - \lambda_\ell \lambda_k},
\]
where \( \lambda_k = T \rho_k + 1 \). We then obtain
\[
\left( \frac{1}{q_\rho} \frac{1 - \lambda_k}{1 - \lambda_k} \right) \frac{1}{\delta} = \frac{T(|\lambda_k|^2 - 1)}{(1 - \lambda_k)(T \delta + 1 - \lambda_k)}.
\]
And also by taking into account the norms relation (2.20) we get

\[ \frac{1}{T \delta + 1 - \lambda_k} \leq \frac{1}{T}, \quad \frac{1}{z - \lambda_k} \leq \frac{1}{T |\lambda_k|^2 - 1}. \]

This can be utilized to show \( J_2 = J_{\delta u2} \).

### 3.4.3 Tracking Error Problem under Control Input Penalty

We can easily extend the result to one with control input penalty, i.e., we consider the following tracking measure:

\[ J_{\delta} := T \sum_{k=0}^{\infty} (\|e(k)\|^2 + |u_w(k)|^2), \quad (3.60) \]

where \( u_w(k) = D^{-1} \{ W_u(\delta) \hat{u}(\delta) \} \). The optimal performance is then provided by

\[ J_\delta^* = \inf_{Q_a \in \mathcal{H}_\infty} \left\| [I + N_a(\hat{Y}_a - Q_a \tilde{M}_a)] \nu_a \delta \right\|_2^2, \quad (3.61) \]

In addition to Assumption 3.7 and 3.8, we make the following assumption.

**Assumption 3.9** \( P(\delta) \) has a pole at \( \delta = 0 \).

We also define an inner-outer factorization such that

\[ N_a(\delta) = \begin{bmatrix} W_u(\delta) M(\delta) \\ N(\delta) \end{bmatrix} = \Theta_i(\delta) \Theta_o(\delta). \quad (3.62) \]

**Theorem 3.8** Suppose that the SIMO plant \( P(\delta) \) given in (2.3) has unstable poles \( \rho_k \) \( (k = 1, \ldots, n_p) \) and \( P_i(\delta) \) has non-minimum phase \( \zeta_{ij} \) \((i = 1, \ldots, m, j = 1, \ldots, n_i)\). Then, under Assumptions 3.7–3.9, the optimal tracking performance under control penalty is given by

\[ J_\delta^* = J_{\delta s}^* + J_{\delta u}^*, \quad (3.63) \]

where

\[ J_{\delta s}^* = J_{\delta s1} + J_{\delta s2}, \]
\[ J_{\delta u}^* = J_{\delta u1} + J_{\delta u2} \]

with

\[ J_{\delta s1} := \sum_{i \in I_a} \nu_i^2 \sum_{j=1}^{n_i} \left( 2 \operatorname{Re} \frac{\zeta_{ij}}{|\zeta_{ij}|^2} + T \right), \]
\[ J_{\delta s2} := \frac{T^2}{2\pi} \sum_{i \in I_a} \nu_i^2 \int_{0}^{\pi/T} \frac{\log \frac{[P_i(0)]^2 + [P_i(\frac{\omega T}{\pi})]^2 + [W_u(\frac{\omega T}{\pi})]^2}{|P_i(\frac{\omega T}{\pi})|^2 + [W_u(\frac{\omega T}{\pi})]^2}}{1 - \cos \omega T} \, d\omega, \]
\[ J_{\delta u_1} := \sum_{i \in I \nu} \nu_{i}^{2} \sum_{k \in I p} \left( \frac{2 \text{Re} \rho_{k}}{|\rho_{k}|^{2}} + T \right), \]
\[ J_{\delta u_2} := \sum_{k, \ell \in I p} \frac{(T|\rho_{k}|^{2} + 2 \text{Re} \rho_{k})(T|\rho_{\ell}|^{2} + 2 \text{Re} \rho_{\ell})}{\hat{q}_{k}\hat{q}_{\ell}\rho_{k}(T\hat{\rho}_{k}\rho_{k} + \rho_{k} + \rho_{\ell})} \times \]
\[ (1 - \Theta\sim(\hat{\rho}_{k})\Theta(0))(1 - \Theta\sim(\rho_{\ell})\Theta(0)), \]
and
\[ q_{k} := \begin{cases} 1 & : \#_{p} = 1 \\ \frac{1}{T} \prod_{\ell \in I p, \ell \neq k} \frac{\rho_{\ell} - \rho_{k}}{T\hat{\rho}_{k}\rho_{k} + \rho_{k} + \rho_{\ell}} & : \#_{p} \geq 2. \end{cases} \]

**Proof.** The proof is similar to that of Theorem 3.6. \( \blacksquare \)

### 3.4.4 Continuity Property

#### Plant Transformation

Consider a continuous-time plant \( G_{c}(s) \) with the following state space representation:
\[
G_{c}(s) = \begin{bmatrix} A_{c} & B_{c} \\ C_{c} & D_{c} \end{bmatrix} = C_{c}(sI - A_{c})^{-1}B_{c} + D_{c}.
\]
Suppose that the sampling interval is \( T \). By using the step invariance transform or the zero-order hold (ZOH), i.e., \( x(t) = x(kT) \), \( kT \leq t < (k + 1)T \), discretizing the plant \( G_{c}(s) \) gives a corresponding discrete-time plant
\[
G_{d}(z) = \begin{bmatrix} A_{d} & B_{d} \\ C_{d} & D_{d} \end{bmatrix} = C_{d}(zI - A_{d})^{-1}B_{d} + D_{d},
\]
where
\[
A_{d} := e^{A_{c}T}, \quad B_{d} := \int_{0}^{T} e^{A_{c}t} \, dt B_{c}.
\]
Subsequently, by using the delta operator described in Section 2.4 we obtain the corresponding delta domain plant \( G_{T}(\delta) \) as follows,
\[
G_{T}(\delta) = \begin{bmatrix} A_{T} & B_{T} \\ C_{T} & D_{T} \end{bmatrix} = C_{T}(\delta I - A_{T})^{-1}B_{T} + D_{c},
\]
where
\[
A_{T} := \frac{e^{A_{c}T} - I}{T}, \quad B_{T} := \frac{1}{T} \int_{0}^{T} e^{A_{c}t} \, dt B_{c}.
\]
As \( T \) tends to zero, it can be readily seen that \( A_{T} \) tends to \( A_{c} \) and \( B_{T} \) tends to \( B_{c} \). Therefore, by taking into account the relation \( \delta = (e^{sT} - 1)/T \), we see that \( P_{T}(\delta) \) tends to \( P_{c}(s) \).
Pole-Zero Transformation

Let \( G_c(s) \) be rational function

\[
G_c(s) = \frac{c(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)},
\]

where \( r := n - m > 0 \). It is known that when a continuous-time plant \( G_c(s) \) is discretized under sampling time \( T \) to \( G_d(z) \) then the poles \( p_k \) \((k = 1, \ldots, n)\) are transformed as

\[
p_k \rightarrow \lambda_k := e^{p_k T},
\]

in which the stability is preserved. There is unfortunately no simple transformation which shows how the zeros of continuous-time plant \( z_i \) \((i = 1, \ldots, m)\) are mapped by sampling. Generically, the transfer function \( G_d(z) \) has \( n-1 \) zeros. For particular values of \( T \), some zeros may go to infinity, or they may be canceled by poles. However, for limiting cases where \( T \) is sufficiently small, there is a useful characterization \cite{2}. As sampling time \( T \rightarrow 0 \), \( m \) zeros of \( G_d(z) \) go to 1 as

\[
z_i \rightarrow \eta_i := e^{z_i T}
\]

and the remaining \( r - 1 \) zeros of \( G_d(z) \) go to the zeros of polynomial

\[
B_r(z) = b_r^1 z^{n-1} + b_r^2 z^{n-2} + \ldots + b_r^n,
\]

where

\[
b_r^k := \sum_{l=1}^{k} (-1)^{k-l} p^r \binom{r+1}{k-1}, \quad k = 1, \ldots, n.
\]

It is important to note that the number of non-minimum phase zeros of \( B_r(z) \) is \( \frac{r-1}{2} \) for odd \( r \) and \( \frac{r}{2} \) or \( \frac{r}{2} - 1 \) for even \( r \). This fact means that in many cases non-minimum phase zeros will appear as the sampling time is decreased even though the continuous-time plant may be minimum phase.

Summarizing, if the continuous-time plant \( G_c(s) \) has unstable poles \( p_k \) \((k = 1, \ldots, n)\) and non-minimum phase zeros \( z_i \) \((i = 1, \ldots, m)\), then if \( T \) is sufficiently small the corresponding discrete-time plant \( G_d(z) \) has unstable poles \( \lambda_k = e^{p_k T} \) \((k = 1, \ldots, n)\) and non-minimum phase zeros \( \eta_i = e^{z_i T} \) \((i = 1, \ldots, m)\) and \( \eta^{*}_i \) \((i = 1, \ldots, m^*)\), where \( m^* \) is either \( \frac{r-1}{2} \) or \( \frac{r}{2} \) or \( \frac{r}{2} - 1 \). Note that the limiting zeros \( \eta^{*}_i \) are all finite. As a direct implication, the corresponding delta domain plant \( G_T(\delta) \) has unstable poles \( p_k := e^{\frac{p_k T}{2}} \) \((k = 1, \ldots, n)\) and non-minimum phase zeros \( \zeta_i = e^{\frac{z_i T}{2}} \) \((i = 1, \ldots, m)\) and \( \zeta^{*}_i = \frac{\eta^{*}_i}{2} \) \((i = 1, \ldots, m^*)\).

Convergence of the Expression

Here we will show the continuity properties of the delta domain expressions, i.e., we demonstrate that delta domain expressions in Theorem 3.8 converge...
3.4 Delta Domain Case

To their continuous-time counterparts in Theorem 3.3 when the sampling time \( T \) tends to zero.

To avoid ambiguity, we denote by

\[
P_c(s) = (P_{c1}(s), P_{c2}(s), \ldots, P_{cm}(s))^T,
\]

the respecting continuous-time plant. Suppose that \( P_c(s) \) has unstable poles \( p_k(k = 1, \ldots, n_p) \) and \( P_{cij}(s) \) has non-minimum phase zeros \( z_{ij}(i = 1, \ldots, m, j = 1, \ldots, n_i) \). Under the zero-order hold operations we obtain the corresponding delta domain plant

\[
P_T(\delta) = (P_{T1}(\delta), P_{T2}(\delta), \ldots, P_{Tm}(\delta))^T,
\]

where \( P_T(\delta) \) has unstable poles \( \rho_k(k = 1, \ldots, n_\rho) \) with \( n_\rho = n_p \), and \( P_{Ti}(\delta) \) has non-minimum phase zeros \( \zeta_{ij}(i = 1, \ldots, m, j = 1, \ldots, n_i) \) and \( \zeta^{\star}_{ij}(i = 1, \ldots, m, j = 1, \ldots, n_i^{\star}) \).

Now we are ready to show the convergence. Recall the index sets \( \mathbb{K}_a, \mathbb{K}_p, \mathbb{K}_{pi} \) introduced in Subsection 3.2.2. From the previous part we know the following pole-zero relationships:

\[
\rho_k = e^{p_k T - 1/T}, \quad k = 1, \ldots, n_p,
\]

\[
\zeta_{ij} = e^{z_{ij} T - 1/T}, \quad i = 1, \ldots, m, j = 1, \ldots, n_i,
\]

\[
\zeta^{\star}_{ij} = \eta_{ij}^{\star} - 1/T, \quad i = 1, \ldots, m, j = 1, \ldots, n_i^{\star},
\]

where \( \eta_{ij}^{\star} \) are the zeros of polynomial \( B_r(z) \) defined in (3.66). Since \( \zeta_{ij} \to z_{ij} \) and \( \zeta_{ij}^{\star} \to \infty \) as \( T \) tends to zero, obviously we have

\[
\lim_{T \to 0} J_{S11} = \sum_{i \in \mathbb{K}_a} \nu_i^2 \sum_{j=1}^{n_i} \frac{2 \text{Re } z_{ij}}{|z_{ij}|^2} =: J_{c1}.
\]

(3.67)

Immediately by fact that \( \rho_k \to p_k \) as \( T \) tends to zero we also have

\[
\lim_{T \to 0} J_{S11} = \sum_{i \in \mathbb{K}_a} \nu_i^2 \sum_{k \in \mathbb{K}_{pi}} \frac{2 \text{Re } p_k}{|p_k|^2} =: J_{c1}.
\]

(3.68)

Next, since \( P_{Ti}(\delta) \to P_{ci}(s) \) and \( W_{Tu}(\delta) \to W_{cu}(s) \) as \( T \) tends to zero, and

\[
\lim_{T \to 0} \frac{T^2}{2(1 - \cos \omega T)} = \frac{1}{\omega^2}, \quad \lim_{T \to 0} \frac{e^{j\omega T} - 1}{T} = j\omega
\]

then we obtain

\[
\lim_{T \to 0} J_{S22} = \frac{1}{\pi} \sum_{i \in \mathbb{K}_a} \nu_i^2 \int_0^\infty \log \left[ \frac{|P_{ci}(0)|^2 \|P_{ci}(j\omega)\|^2 + |W_{cu}(j\omega)|^2}{\|P_{ci}(0)\|^2 |P_{ci}(j\omega)|^2} \right] \frac{d\omega}{\omega^2} =: J_{c2}.
\]

(3.69)
We show the convergence of $J_{\delta u2}$ part by part. Since
\[
\lim_{T \to 0} q_k = \prod_{\ell \in K_p, \ell \neq k} \frac{p_\ell - p_k}{p_\ell + p_k} =: \sigma_k,
\]
we readily have
\[
\lim_{T \to 0} \left( T|p_k|^2 + 2 \text{Re} p_k(T|p_\ell|^2 + 2 \text{Re} p_\ell) \right) \bar{q}_k q_\ell p_{\ell k} p_{\ell} \left( \bar{p}_k p_\ell + p_k + p_\ell \right) = \frac{4 \text{Re} p_k \text{Re} p_\ell}{\sigma_k \sigma_\ell p_k p_\ell (p_k + p_\ell)}.
\]
Now we deal with terms that contain the inner factor $\Theta_{T}(p_k)$. First note that in delta domain we define $\Theta_{\sim T}(p_k) := \Theta_{T}(\bar{T}_p p_k (p_k + 1))$. Hence, it is clear that $\frac{-p_k}{p_k + 1} \to -p_k$ as $T$ tends to zero. Recall the inner-outer factorization such that
\[
\begin{bmatrix}
M_T(\delta)
N_T(\delta)
\end{bmatrix} = \Theta_{T}(\delta) \Theta_{T \circ (\delta)}.
\]
Here we set $W_c(u(s)) = 1$ without loss of generality. Note that we have the following representation:
\[
\begin{bmatrix}
M_T(\delta)
N_T(\delta)
\end{bmatrix} = \begin{bmatrix}
A_T + B_T F_T & B_T \\
F_T & C + D_c F_T
\end{bmatrix},
\]
where matrix $F_T$ is chosen such that $(A_T + B_T F_T)$ is stable. Since all the matrices converge to the continuous-time counterparts as $T$ tends to zero, then it is sufficient to conclude that $\Theta_{T}(\delta) \to \Theta_{ci}(s)$ as $T \to 0$. Therefore,
\[
\lim_{T \to 0} J_{\delta u2} = \sum_{k, \ell \in K_p} 4 \text{Re} p_k \text{Re} p_\ell \frac{(1 - \Theta_{ci}^{-1}(p_k) \Theta_{ci}(0))(1 - \Theta_{ci}^{-1}(p_\ell) \Theta_{ci}(0))}{(p_k + p_\ell) p_k p_\ell \sigma_k \sigma_\ell} =: J_{cu2},
\]
where we define $\Theta_{ci}^{-1}(p_k) := \Theta_{ci}^{T}(\bar{p}_k)$.

Overall, (3.67)–(3.70) show the continuity property:
\[
\lim_{T \to 0} J_{\delta}^* = J_{*}^c,
\]
where $J_{*}^c$ is given in Theorem 3.3. In other words, we completely recover the continuous-time expression from the delta domain expression standpoint by making the sampling time approaches zero.

### 3.5 Delay-time Case

It is well-known that the time delays generally degrade the optimal tracking performance in much the same manner as non-minimum phase zeros [14, 58]. In this section, we exploit the delta domain expression to re-derive the
analytical closed-form expression of the optimal tracking performance for delay-time systems.

We consider the unity feedback control system depicted in Fig. 3.1, where $P$ is the SIMO plant to be controlled with delay in the input port:

$$P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \\ \vdots \\ P_m(s) \end{bmatrix} e^{-\tau s},$$  \hspace{1cm} (3.72)

where $P_i (i = 1, \ldots, m)$ are scalar transfer functions and $\tau \geq 0$ is the time-delay. We minimize the performance index given in (3.9), that is

$$J_\tau := \int_0^{\infty} (\|e(t)\|^2 + |u_w(t)|^2) \, dt.$$

Under the zero-order hold operations of sampling time $T$, we can obtain the delta domain counterpart of the plant (3.72) as follows

$$P(\delta) = \frac{1}{(T\delta + 1)^{\tau/T+1}} \begin{bmatrix} P_1(\delta) \\ P_2(\delta) \\ \vdots \\ P_m(\delta) \end{bmatrix}.$$  \hspace{1cm} (3.73)

Note that the relative degree $\tau/T$ is resulted from the discretization of the delay part $e^{-\tau s}$. We will also receive additional relative degree 1 contributed by $P_i(s) (i = 1, \ldots, m)$ provided that $T$ is sufficiently small. We denote by $\rho_k \in \mathbb{D}_T^+ (k = 1, \ldots, n_p)$ the unstable poles of $P(\delta)$ and by $\zeta_{ij} \in \mathbb{D}_T^+ (i = 1, \ldots, m, \ j = 1, \ldots, n_i)$ the non-minimum phase zeros of $P_i(\delta)$.

**Theorem 3.9** Suppose that the delay-time plant $P(s)$ given in (3.72) has unstable poles $p_k \in \mathbb{C}_+ (k = 1, \ldots, n_p)$ and $P_i(s)$ has non-minimum phase zeros $z_{ij} \in \mathbb{C}_+ (i = 1, \ldots, m, \ j = 1, \ldots, n_i)$. Then, the optimal tracking performance $J_\tau^*$ is given by

$$J_\tau^* = J_{\tau s1} + J_{\tau s2} + J_{\tau u1} + J_{\tau u2},$$  \hspace{1cm} (3.74)

where

$$J_{\tau s1} := \tau + \sum_{i \in I} \sum_{j=1}^{n_i} \frac{2 \Re z_{ij}}{|z_{ij}|^2}, \quad J_{\tau s2} := J_{cs2}, \quad J_{\tau u1} := J_{cu1}, \quad J_{\tau u2} := J_{cu2}$$

with $J_{cs2}, J_{cu1},$ and $J_{cu2}$ have the same expressions as those given by Theorem 3.3.

**Proof.** Since $P(\delta)$ has extra non-minimum phase zeros at infinity with multiplicity $\tau/T + 1$, then based on the continuity property of Theorem 3.8 we obtain
3 Tracking Performance Limitations

\[ J_{rs1} = \lim_{T \to 0} \sum_{i \in I} \nu_i^2 \sum_{j=1}^{n_i+\tau/T+1} \left( \frac{2 \text{Re} \zeta_{ij}}{|\zeta_{ij}|^2} + T \right) \]

\[ = \lim_{T \to 0} \sum_{i \in I} \nu_i^2 \sum_{j=1}^{\tau/T+1} T + \lim_{T \to 0} \sum_{i \in I} \nu_i^2 \sum_{j=1}^{n_i} \left( \frac{2 \text{Re} \zeta_{ij}}{|\zeta_{ij}|^2} + T \right) \]

\[ = \tau + \sum_{i \in I} \nu_i^2 \sum_{j=1}^{n_i} \frac{2 \text{Re} z_{ij}}{|z_{ij}|^2}. \]

Note that \( \nu \) is a unitary vector. The proof for other terms follow the continuity property of Theorem 3.8.

3.6 Sampled-data Case

Modern control systems are almost always implemented in a digital computer. It is thus important to have an appreciation of the impact of implementing a particular control law in digital form. Studies in this subject include the frequency domain analysis for sampled-data systems, in which one of the work introduced the use of lifting techniques. By lifting, a signal valued in a finite dimensional space is bijectively mapped onto a signal valued in infinite dimensional spaces. Attractively, by this transformation it has become possible to view sampled-data system as an LTI discrete-time system with built-in inter-sample behavior.

Working with sampled-data system is naturally harder than that with continuous-time system or discrete-time system. In sampled-data feedback control problems we have a continuous-time plant and want to design a stabilizing discrete-time controller.

Problems concerning on the fundamental design limitations in sampled-data control systems have been widely investigated since last decade [26, 29, 47] and one pertains to the tracking performance limitations of stable plant is recently studied in [15], which gives the analytical closed-form expression of the optimal tracking performance. In this paper, the problem of tracking a step reference signal using sampled-data control systems is studied by adopting a frequency domain lifting technique. Consequently, the problem is then reduced to one of tracking in equivalent discrete-time system, in which the solution of the latter problem is readily available [58].

For scalar stable case, the optimal tracking performance is governed by following expression [15]:

\[ J_{sd}^* = \sum_{k=1}^{\infty} \frac{2}{2k} + T \sum_{k=1}^{\infty} \frac{2k+1}{2k-1} + \]

\[ \frac{T^2}{2\pi} \int_{0}^{\pi/T} \frac{d\omega}{1 - \cos \omega T}, \quad (3.75) \]
where $z_k$ and $\varsigma_k$ are respectively the non-minimum phase zeros of $P_c(s)$ and a product of ZOH equivalent discretized transfer functions defined by $P_d(z) := z \mathcal{Z} \{P_m(s)H(s)\} \mathcal{Z} \{F(s)\}$. Here, $H$ is the zero-order hold of sampling time $T$, $F$ is anti-aliasing filter, $P_m$ is minimum phase part of $P_c$, and $P_k$ is the harmonics of $P_c$, i.e., $P_k(s) = P_c(s + 2\pi jk/T)$.

It is shown in (3.75) that the optimal performance is explicitly determined by the sampling time, the non-minimum phase zeros of the continuous-time plant, the non-minimum phase zeros of the discretized plant, and the aliasing effect incurred by the sampling-hold operations.

Our approach presented in this section is completely different. Instead of compute the exact value of the optimal tracking performance as did in [15], we employ an approximation approach by implementing fast sampling technique. We benefit our preceding result on tracking performance problem of SIMO discrete-time system. By this approach, we can extend the tracking error problem to unstable case and possibly with control input penalty.

We consider the generic setup of a single-input single-output (SISO) sampled-data feedback control system depicted in Fig. 3.10, where $P_c(s)$ represents the continuous-time plant and $K_d(z)$ the discrete-time stabilizing controller. The signal $r$ is the reference input and considered to be a unit step function. Signals $u$, $y$, $e := r - y$ denote the control input, measurement output, and error response, respectively. While, $e_k$ and $u_k$ represent digital signals relate to $e$ and $u$ conducted by the sampler $S$ and the zero-order hold $H$ of sampling time $T$. The set of all stabilizing controllers is then defined by

$$K_{sd} := \{K_d(z) : K_d(z) \text{ stabilizes } P_c(s)\}. \quad (3.76)$$

We want to minimize the following tracking measure with respect to all stabilizing controllers:

$$J_{sd} = \int_0^\infty (|e(t)|^2 + |u_w(t)|^2) \, dt, \quad (3.77)$$

where $u_w(t) = W_u \mathcal{L}^{-1}\{\hat{u}(s)\}$. Here we consider a real constant weighting function $W_u$ for simplicity since it will also give a constant under sampling.

We implement Assumptions 3.1 and 3.3, i.e., we assume that $N_c(0) \neq 0$ and $P_c(s)$ has a pole at $s = 0$.

### 3.6.1 Fast Sampling

Under the fast sampling technique, we embed a fast sampler $S_f$ with sampling time $T/N$ at the reference input and the plant output, from which
we subdivide the $k$-th sampling interval $[kT, (k + 1)T)$ into $N$ subintervals $[kT + \frac{i}{N}, kT + \frac{i+1}{N}T), i = 0, 1, \ldots, N - 1$. By this process, the feedback control setup of Fig. 3.10 can be approximated by that of Fig. 3.11. Equivalently, Fig. 3.11 can be re-arranged such that it becomes Fig. 3.12. In these figures we denote

$\mathbf{r}_k := \begin{bmatrix} r_k \\ r_k \\ \vdots \\ r_k \end{bmatrix}$, $\mathbf{y}_k := \begin{bmatrix} y_{k0} \\ y_{k1} \\ \vdots \\ y_{kN-1} \end{bmatrix}$,

where $r_k$ is a discrete-time unit step function and $y_{ki} = y(kT + \frac{i}{N}T)$, for $i = 0, 1, \ldots, N - 1$.

Suppose that the transfer functions of the continuous-time plant $P_c(s)$ and its discretized plant $P_d(z)$ are determined by

$P_c(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $P_d(z) = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$,

where

$A_d = e^{AT}$, $B_d = \int_0^T e^{At} B \, dt$, $C_d = C$, $D_d = D$.

It is not difficult to verify that

$y_{ki} = C_d e^{AT} x(kT + \frac{i}{N}T) + \left( C_d \int_0^{\frac{i}{N}T} e^{At} B \, dt + D_d \right) u(kT + \frac{i}{N}T)$

for $i = 0, 1, \ldots, N - 1$. The transfer functions from $u_k$ to $y_{ki}$, denoted by $P_{t,i}$, are then determined by

$P_{t,i}(z) = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$, \hspace{1cm} (3.78)
\[ C_d(t) = C_d e^{At} T, \quad D_d(t) = C_d \int_0^T e^{At} B \, dt + D_d. \]

Obviously, \( P_{i0}(z) = P_0(z) \) and we can see that \( P_{i}(i = 0, 1, \ldots, N - 1) \) have the same set of unstable poles. Furthermore, we define

\[ P_i(z) = (P_{i0}(z), P_{i1}(z), \ldots, P_{iN-1}(z))^T. \]  \hfill (3.79)

To solve the problem we implement the plant augmentation strategy initially introduced in [31]. Let define the augmented plant

\[ P_a(s) := \begin{bmatrix} W_u & P_c(s) \end{bmatrix}, \]

from which by fast sampling procedure we can approximate the performance index (3.77) by

\[ J_f := \frac{T}{N} \sum_{k=0}^{\infty} \| r_{ak} - y_{ak} \|^2, \]  \hfill (3.80)

where

\[ r_{ak} = \begin{bmatrix} 0 \end{bmatrix}, \quad y_{ak} = \begin{bmatrix} u_{wk} \\ y_k \end{bmatrix} \]

with \( 0_N := (0, \ldots, 0)^T \in \mathbb{R}^N \) and \( u_{wk} := (u_{wk0}, \ldots, u_{wkN-1})^T \). We put a factor of \( \frac{T}{N} \) as implication of the sampling and hold operations. Furthermore, by Parseval’s identity we obtain

\[ J_f = \frac{T}{N} \left\| \begin{bmatrix} 0 \\ 1_N \end{bmatrix} \hat{r}_k - \begin{bmatrix} \sqrt{NW} \\ P_i \end{bmatrix} \hat{u}_k \right\|^2, \]  \hfill (3.81)

where \( 1_N := (1, \ldots, 1)^T \in \mathbb{R}^N \). Note that originally we fast-sample the signal \( u_w \) such that we obtain the transfer functions \( (W, \ldots, W)^T \) of \( N \)-tuple. Since the sampling points are all constant we represent them only by single point \( \sqrt{NW} \).

### 3.6.2 Approximation of the Optimal Performance

Let the coprime factorization of \( P_{i0} \) is given by

\[ P_{i0}(z) = N_{i0}(z) M_{i0}^{-1}(z), \]

where

\[ M_{i0}(z) = \prod_{k=1}^{n_{i0}} \frac{z - \lambda_k}{1 - \lambda_k z} \]

with \( \lambda_k (k = 1, \ldots, n_{i0}) \) are the unstable poles of \( P_{i0}(z) \). Since \( P_{i}(i = 0, \ldots, N - 1) \) have only common unstable poles then the coprime factorization of \( P_i \) is given by
\[ P_t(z) = N_t(z)M_{10}^{-1}(z), \]
where \( N_t = (N_{t0}, N_{t1}, \ldots, N_{tn-1})^T \). Youla parameterization (2.8) tells that the stabilizing digital controller is parameterized by
\[ K_d = \frac{Y_{t0} - M_{t0}Q_t}{N_{t0}Q_t - X_{t0}}, \]
where \( Q_t \in \mathbb{R} \mathcal{H}_\infty \) is a scalar free parameter. Since \( \hat{e}_k = M_{t0}(X_{t0} - N_{t0}Q_t)\hat{r}_k \), it yields \( \hat{u}_k = -M_{t0}(Y_{t0} - N_{t0}Q_t)\hat{r}_k \) and furthermore \( \hat{y}_k = -N_{t0}(Y_{t0} - M_{t0}Q_t)\hat{r}_k \). Consequently the minimum value of (3.81) is given by
\[ J^*_{f} = T\inf_{Q_t \in \mathbb{R} \mathcal{H}_\infty} \| \nu_t + N_{fa}(Y_{fa} - Q_{fa}M_{fa}) \|_2^2, \]
where \( \nu_t = (0, 1, \ldots, 1)^T \in \mathbb{R}^{N+1} \) and
\[ N_{fa}(z) = \begin{bmatrix} \sqrt{N_{wa}M_{fa}(z)} \\ N_{t}(z) \end{bmatrix}. \]
Furthermore, we can write (3.82) as
\[ J^*_{f} = T\inf_{Q_{fa} \in \mathbb{R} \mathcal{H}_\infty} \| [I + N_{fa}(Y_{fa} - Q_{fa}M_{fa})]\nu_t \|_2^2, \]
where \( Q_{fa} = (0^T_{N}, Q_{t}) \) and \( Y_{fa} = (0^T_{N}, Y_{fa}) \).

Fortunately, the last expression of \( J^*_{f} \) is coincident with that of the optimal tracking performance for SIMO discrete-time case \( J^*_{d} \) in (3.49) whenever \( \nu_a = \nu_t \). The only difference is that \( M_{fa} \) is scalar, but \( M_{fa} \) is square. Hence, by defining an inner-outer factorization such that \( N_{fa} = \Theta_i\Theta_o \) we can invoke the further derivation of (3.49) to produce
\[ J^*_{f} = \frac{T}{N}\inf_{Q_{fa} \in \mathbb{R} \mathcal{H}_\infty} \| (\Theta_i^{-1} - (1))\nu_t \|_2^2 \]
\[ \quad + \| (I - \Theta_\alpha(\Theta_i^{-1}))\nu_t \|_2^2 \]
\[ \quad + \inf_{Q_{fa} \in \mathbb{R} \mathcal{H}_\infty} \| \Theta_i^{-1}(1) + \Theta_\alpha(Y_{fa} - Q_{fa}M_{fa})\nu_t \|_2^2 \].

**Theorem 3.10** Consider the sampled-data system depicted in Fig. 3.10 with an SISO plant \( P_c(s) \). Let \( \eta_{ij} \) (\( i = 0, \ldots, N - 1, j = 1, \ldots, n_i \)) be the non-minimum phase zeros of \( P_t(z) \) and \( \lambda_k \) (\( k = 1, \ldots, n_\lambda \)) be the unstable poles of \( P_t(z) \). Then the approximation value of the optimal tracking error performance is given by
\[ J^*_{f} = J_{f1} + J_{f2} + J_{fu2}, \]
where
from which we simplify

\[ J_{ts1} := \frac{T}{N} \sum_{i=0}^{N-1} \sum_{j=1}^{n_i} |\theta_{ij}|^2 - 1 |\theta_{ij} - 1|^2, \]

\[ J_{ts2} := \frac{T}{2\pi N} \sum_{i=0}^{N-1} \int_{0}^{\pi} \log \left[ \frac{|P_{ic}(1)|^2 ||P_{ic}(e^{j\theta})||^2 + NW_u^2}{|P_{ic}(e^{j\theta})|^2} \right] \frac{d\theta}{1 - \cos \theta}, \]

\[ J_{ts2} := \frac{T}{N} \sum_{k,\ell=1}^{n_s} \left| \frac{[\lambda_k^2 - 1)([\lambda_\ell^2 - 1)(1 - \Theta^{-1}_c(\lambda_k)\Theta(1)))(1 - \Theta^{-1}_c(\lambda_\ell)\Theta(1))}{\pi \sum_{n=0}^\pi \lambda_n^2} \right|^2, \]

with

\[ h_k := \begin{cases} 1 & ; n_\lambda = 1, \\
\prod_{j \neq k} \lambda_j - \lambda_k & ; n_\lambda \geq 2. \end{cases} \]

**Proof.** If \( P_c(s) \) has unstable poles \( p_k \) \((k = 1, \ldots, n_p)\) then the discretized plant \( P_c \) will have only common unstable poles \( \lambda_k \) \((k = 1, \ldots, n_\lambda)\), where \( \lambda_k = e^{p_k T} \) and \( n_p = n_\lambda \). Consequently, \( J_{ts2} \) is non-negative since \( M_{ts}(\lambda_k)n_\ell = 0 \), but \( J_{ts1} = 0 \). Note that if \( P_c(s) \) is marginally stable then \( J_{ts2} = 0 \), and hence we can compute \( J^* \) without using \( \Theta^{-1}_c \).

**Example 3.6** Having a sampled-data feedback control system in Fig. 3.10, we consider the following SISO continuous-time plant:

\[ P_c(s) = \frac{s - x}{s(s + 1)}, \quad x > 0. \]

Note that \( P_c(s) \) is marginally stable and has a non-minimum phase zero at \( s = x \). It is not difficult to verify that

\[ A_d = \begin{bmatrix} e^{-T} & 0 \\
1 - e^{-T} & 1 \end{bmatrix}, \]

\[ B_d = \begin{bmatrix} 1 - e^{-T} \\
T + e^{-T} - 1 \end{bmatrix}, \]

\[ C_d = (1 + x)e^{-\frac{T}{2}} - x, \quad -x, \]

\[ D_d = 1 + x(1 - iT/N) - (1 + x)e^{-\frac{T}{2}}. \]

Suppose that \( n_{s_1}(z) \) is the numerator of \( P_s(z) \). Then, \( n_{s_1}(1) = x(1 - e^{-T})(1 - 2e^{-T} - T) \) for all \( i \). Consequently,

\[ \frac{|P_{s}(1)|^2}{||P_{s}(1)||^2} = \frac{|n_{s_1}(1)|^2}{\sum_{i=0}^{N-1} |n_{s_1}(1)|^2} = \frac{1}{N}, \]

from which we simplify

\[ J_{ts2} = \frac{T}{2\pi N} \sum_{i=0}^{N-1} \int_{0}^{\pi} \log \left[ \frac{1}{N} \frac{||P_{s}(e^{j\theta})||^2 + NW_u^2}{||P_{s}(e^{j\theta})||^2} \right] \frac{d\theta}{1 - \cos \theta}. \]
First we consider a case without input penalty, i.e., $W_u = 0$. We compute the optimal tracking performance for different pairs of $(T, N)$: $(0.1\text{sec.}, 30)$ and $(0.01\text{sec.}, 3)$ by using Theorem 3.10. We also compute the exact value by using [15, Theorem 1]. Fig. 3.13, which plots the optimal performance for $x$ from 1 to 3, shows that we approximate the exact results well. Particularly if the sampling time $T$ is small, $N$ can be made small. Second we consider nonzero $W_u$. We select $W_u = \{5, 3, 1\} \times 10^{-5}$ and compute the optimal performance for $T = 0.01\text{sec.}$ and $N = 3$. Fig. 3.14 shows that the results converge to those of the first case as $W_u$ gets smaller.
3.7 Summary

We have examined the $\mathcal{H}_2$ tracking performance problem in SIMO LTI feedback control system. We have formulated and solved the tracking problem in minimizing the tracking error under control input penalty with respect to a step reference input. We characterize and quantify the tracking performance limitations may arise in terms of the dynamics and structure of the plant. In other words we derive the analytical closed-form expression of the optimal performance.

We have made few corrections toward the existing results of continuous-time case and have presented a new result for discrete-time counterpart. Application of the bilinear transformation enables us to derive the discrete-time results straightforwardly. We have also reformulated and resolved the problem in the delta domain, from which we can recover the continuous-time results by approaching the sampling time to zero. Additionally, we have provided the approximation of the optimal performance in sampled-data feedback control system by applying the fast sampling technique.

In general, our results show that the optimal tracking performance is properly characterized by the plant’s non-minimum phase zeros and unstable poles, plant gain, and the direction of the reference input. The derived expressions are not complete yet in the sense that they include the inner factor.
Regulation Performance Limitations

In this chapter, we investigate the regulation properties pertaining to single-input multiple-output (SIMO) linear time-invariant (LTI) systems. We provide analytical closed-form expressions of the best achievable $\mathcal{H}_2$ optimal regulation performances against impulsive disturbance inputs for unstable and non-minimum phase continuous-time and discrete-time systems. We also modify the latter results by means the delta operator and show the continuity property. In this step, we can also solve the regulation problem of delay-time systems.

We here mainly focus on the two-kind of optimal regulation problem, namely the energy regulation problem and the output regulation problem. In the former problem, the regulation performance is measured by minimizing the energy of control input, while in the latter problem, by minimizing the energy of control input jointly with the energy of measurement output. Results on $\mathcal{H}_2$ energy regulation problem of continuous-time case can be found in [31]. Equivalent results in SISO systems but articulated in term of signal-to-noise ratio constrained channels are found in [4, 42]. Meanwhile, results on $\mathcal{H}_2$ output regulation problem of minimum phase SISO/MIMO systems are presented in [14]. In the present work we complete the result on output regulation problem for continuous-time case by considering non-minimum phase system. While in the discrete-time case we provide new results.

Applications of our results on the regulation problem of three-disk torsional system and magnetic bearing system can be found in Chapter 5.

4.1 Regulation Performance Problem

This section is devoted to the formulation of the $\mathcal{H}_2$ optimal regulation problem. We consider the feedback control setup depicted by Fig. 4.1. In this setup the functions $W_s$ and $W_y$ respectively denote the weighting for the sensitivity reduction and that for the disturbance attenuation, which are stable and minimum phase. We note that in regulation problem the output responses
are solely resulted from the disturbance input signal \( d \), which is an impulse function. In some sense we can say that the regulation problem is the dual of the tracking problem. In regulation problem the disturbance signal enters the system through the plant input, while in tracking problem the reference signal commands the system through the plant output.

Our primary objective in this work is not on how to find an optimal controller \( K \) which stabilizes the feedback and regulate the system output to zero. Rather, we want to relate the optimal regulation performance with the characteristics of the plant \( P \).

4.1.1 Integral Formulae

We have a number of integral formulae which play important roles in our subsequent derivation.

**Lemma 4.1 (Poisson-Jensen Formula)** Let \( f \) is analytic in \( \mathbb{D}^c \) and \( d_i \) \((i = 1, \ldots, n_d)\) be the zeros of \( f \) in \( \overline{\mathbb{D}}^c \), counting their multiplicities. If \( z \in \overline{\mathbb{D}}^c \) and \( f(z) \neq 0 \), then

\[
\log |f(z)| = \frac{1}{\pi} \int_0^\pi \text{Re} \left( \frac{ze^{i\theta} + 1}{ze^{i\theta} - 1} \right) \log |f(e^{i\theta})| \, d\theta - \sum_{i=1}^{n_d} \log \left| 1 - \frac{d_i z}{z - d_i} \right|.
\]

(4.1)

**Proof.** The Poisson-Jensen formula can be found in many standard books on complex analysis. See for instance [20].

**Lemma 4.2** Let \( g \) is analytic in \( \mathbb{C}_+ \) and \( c_i \) \((i = 1, \ldots, n_c)\) be the zeros of \( g \) in \( \mathbb{C}_+ \), counting their multiplicities. If \( s \in \mathbb{C}_+ \) and \( g(s) \neq 0 \), then

\[
\log |g(s)| = \frac{2}{\pi} \int_0^\infty \text{Re} \left( \frac{1 + js\omega}{s + j\omega} \right) \log |g(j\omega)| \, d\omega - \sum_{i=1}^{n_c} \log \left| \frac{e_i + s}{e_i - s} \right|.
\]

(4.2)

**Proof.** This is the continuous-time version of the Poisson-Jensen formula. Perform the bilinear transformation over Lemma 4.1 to prove it.  

**Lemma 4.3 (Bode’s Attenuation Integral Formula)** Let \( f(s) \in \mathbb{F} \) and denote \( f(j\omega) = f_1(\omega) + jf_2(\omega) \). Suppose that \( f(s) \) is conjugate symmetric, i.e., \( f(s) = \overline{f(\bar{s})} \). Then whenever \( f(\infty) \) exists,

\[
\lim_{s \to \infty} s[f(s) - f(\infty)] = \frac{1}{\pi} \int_{-\infty}^{\infty} [f_1(\omega) - f_1(\infty)] \, d\omega.
\]

(4.3)
4.1 Regulation Performance Problem

Proof. See [53, pp. 49].

Lemma 4.4 Consider a conjugate symmetric function \( f(s) \). Suppose that \( f(s) \) is analytic and has no zero in \( \mathbb{C}_+ \). Then provided that \( f(\infty) \neq 0 \),

\[
\lim_{s \to \infty} s \log \frac{f(s)}{f(\infty)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left| \frac{f(j\omega)}{f(\infty)} \right| d\omega.
\] (4.4)

Proof. An immediate consequence of Lemma 4.3.

Lemma 4.5 If \( f(z) \in \mathbb{R}H_\infty \), then

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \text{Re} f(e^{j\theta}) d\theta = f(\infty).
\] (4.5)

Proof. Suppose that

\[
f(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0},
\]

is stable transfer function, where \( a_k, b_k \in \mathbb{R} (k = 0, 1, \ldots, n) \) and \( b_n \neq 0 \). We may write

\[
f(z) = \frac{a_n}{b_n} + \tilde{f}(z),
\]

where

\[
\tilde{f}(z) = \frac{c_n z^{n-1} + \ldots + c_1 z + c_0}{(z - p_n)(z - p_{n-1}) \ldots (z - p_1)(z - p_0)},
\]

with

\[
c_k = a_k - \frac{a_n b_k}{b_n}, \quad k = 0, 1, \ldots, n - 1,
\]

and \( p_k \) are the poles of \( f \) which lay inside the unit disk, i.e., \( p_k \in \mathbb{D} \), for \( k = 0, 1, \ldots, n \). Hence,

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \text{Re} f(e^{j\theta}) d\theta = \frac{2a_n}{b_n} + \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Re} \tilde{f}(e^{j\theta}) d\theta.
\]

By Cauchy integral,

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(e^{j\theta}) d\theta = -\frac{j}{\pi} \int_{\mathbb{D}} g(z) dz,
\]

where \( g(z) = \tilde{f}(z)/z \). Residue theorem gives

\[
-\frac{j}{\pi} \int_{\mathbb{D}} g(z) dz = 2 \left( \text{Res} g(0) + \sum_{k=0}^{n} \text{Res} g(p_k) \right) = -2\pi \text{Res} g(\infty) = 0.
\]

The proof is complete by fact that \( f(\infty) = a_n/b_n \).
4.1.2 Energy Regulation Problem

We here consider the energy regulation problem, in which the regulation performance is measured by the energy of the control input \( u \). For a given impulse signal \( d \) we define the regulation performance as

\[
E_c := \int_0^\infty |u(t)|^2 \, dt, 
\]

\[
E_d := \sum_{k=0}^\infty |u(k)|^2, \tag{4.6}
\]

for continuous-time and discrete-time cases, respectively. From Parseval identity we can deduce that the best achievable regulation performances by all stabilizing controllers in set \( K \) are given by

\[
E^*_c = \inf_{K \in K} \| K(s)S_o(s)P(s)\hat{d}(s) \|^2_2, 
\]

\[
E^*_d = \inf_{K \in K} \| K(z)S_o(z)P(z)\hat{d}(z) \|^2_2, 
\]

where \( S_o \) is the output sensitivity function defined in (3.2). By considering Bezout identity (2.5) and Youla parameterization (2.8) and by noting that \( S_o = X\tilde{M} - NQ\tilde{M} \), \( \hat{d} = 1 \), the optimal performance can further be expressed as

\[
E^* = \inf_{Q \in \mathcal{H}_\infty} \| Y\tilde{N} - MQ\tilde{N} \|^2_2, \tag{4.8}
\]

where \( E^* \) stands either \( E^*_c \) for continuous-time case or \( E^*_d \) for discrete-time case.

4.1.3 Output Regulation Problem

An extension of the preceding problem can be made by considering a minimization problem of the energy of the measurement output simultaneously with that of the control input. In the present work we consider a more general problem, i.e., we consider the following performance index

\[
E_c := \int_0^\infty \left( \| y_w(t) \|^2 + |u(t)|^2 \right) \, dt, 
\]

\[
E_d := \sum_{k=0}^\infty \left( \| y_w(k) \|^2 + |u(k)|^2 \right), \tag{4.9}
\]

where the signal \( y_w \) is the weighted regulation output, i.e.,

\[
y_w(t) = \begin{bmatrix} y_{w1}(t) \\ y_{w2}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{L}^{-1}\{ W_s(s)\hat{u}(s) + \hat{d}(s) \} \\ \mathcal{L}^{-1}\{ W_y(s)\hat{y}(s) \} \end{bmatrix},
\]

\[
y_w(k) = \begin{bmatrix} y_{w1}(k) \\ y_{w2}(k) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}^{-1}\{ W_s(z)\hat{u}(z) + \hat{d}(z) \} \\ \mathcal{Z}^{-1}\{ W_y(z)\hat{y}(z) \} \end{bmatrix}.
\]
Note that the first and second elements of $y_w$ account the sensitivity reduction and the disturbance attenuation, respectively. If $W_s = 0$ and $W_y = 0$ then the problem reduces to an energy regulation one, which is discussed in the previous subsection.

It follows from the well-known Parseval identity that

$$E = \|\hat{y}_{w1}\|^2_2 + \|\hat{y}_{w2}\|^2_2 + \|\hat{u}\|^2_2$$

holds, where $E$ stands either $E_c$ for the continuous-time case or $E_d$ for the discrete-time case. Then, it is immediate to obtain that

$$E = \|W_s S_i \hat{d}\|^2_2 + \|W_y S_o \hat{d}\|^2_2 + \|K S_o \hat{d}\|^2_2,$$  \hspace{2cm} (4.11)

$$E = \|W_s (1 - K S_o P) \hat{d}\|^2_2 + \|W_y S_o P \hat{d}\|^2_2 + \|K S_o P \hat{d}\|^2_2,$$  \hspace{2cm} (4.12)

where $S_i$ and $S_o$ are the input and output sensitivity functions defined in (3.1) and (3.2), respectively. From (4.11) it is clear that we consider a more general case of regulation problem, where we minimized the regulation output under sensitivity and complementarity sensitivity constraints. Furthermore, Bezout identity (2.5) and Youla parameterization (2.8) provide

$$E = \|W_s (1 + Y \tilde{N} - MQ \tilde{N})\|^2_2 + \|W_y (X \tilde{N} - N Q \tilde{N})\|^2_2 + \|Y \tilde{N} - MQ \tilde{N}\|^2_2,$$

from which we then want to determine the best achievable regulation performance with respect to all stabilizing controllers expressed in the following three-block optimal control problem:

$$E^* = \inf_{Q \in \mathbb{R}^{H_{\infty}}} \left\| \begin{bmatrix} W_s (1 + Y \tilde{N} - MQ \tilde{N}) \\ W_y (X \tilde{N} - N Q \tilde{N}) \\ Y \tilde{N} - MQ \tilde{N} \end{bmatrix} \right\|^2.$$

### 4.2 Continuous-time Case

In this section we provide the analytical closed-form expressions of the optimal regulation performances for continuous-time system.

Recall the coprime factorizations of the plant $P(s)$ given in (2.4). Without loss of generality we may fix the scalar transfer function $M(s)$ as

$$M(s) := \prod_{k=1}^{n_p} \frac{s - p_k}{s + \bar{p}_k},$$  \hspace{2cm} (4.14)

where $p_k$ ($k = 1, \ldots, n_p$) are the unstable poles of $P(s)$. It is useful to point-out that $M(\infty) = 1$. Next, let introduce the following index set:

$$N_z := \{ i : \tilde{N}(z_i) = 0, \ z_i \in \mathbb{C}_+ \}.$$  \hspace{2cm} (4.15)

Note that $N_z$ contains the index set of all common non-minimum phase zeros of $P(s)$, counting their multiplicities.
4.2.1 Energy Regulation Problem

Result on the energy regulation problem of SIMO continuous-time system can be found in [31]. We present the result here and establish its proof. The following theorem gives the minimal value of (4.6), i.e.,

\[ E^*_c = \inf_{K \in \mathcal{K}} \int_0^\infty |u(t)|^2 dt, \]

which is further expressed by (4.8).

**Theorem 4.1 ([31])** Suppose that the plant \( P(s) \) given in (2.4) has common non-minimum phase zeros \( z_i \ (i \in \mathbb{N}_z) \) and unstable poles \( p_k \ (k = 1, \ldots, n_p) \). Then the optimal energy regulation performance is given by

\[ E^*_c = E_{cm} + E_{cn}, \tag{4.16} \]

where

\[ E_{cm} := 2 \sum_{k=1}^{n_p} p_k, \]
\[ E_{cn} := \sum_{i,j \in \mathbb{N}_z} 4 \text{Re} z_i \text{Re} z_j \frac{a_i a_j (\bar{z}_i + z_j)}{\bar{a}_i a_j} \]

with

\[ a_i := \begin{cases} 1 ; & \#\mathbb{N}_z = 1 \\ \prod_{j \in \mathbb{N}_z, j \neq i} \frac{z_j - z_i}{z_j + z_i} ; & \#\mathbb{N}_z \geq 2 \end{cases}, \]
\[ \alpha_i := 1 - \prod_{k=1}^{n_p} \frac{z_i + p_k}{z_i - p_k}. \]

**Proof.** See Appendix A.2. \( \square \)

Theorem 4.1, which is valid for SISO and SIMO systems, shows that unstable poles and non-minimum phase zeros of the plant completely characterize the optimal energy regulation performance. Note that only the common non-minimum phase zero gives an effect, otherwise \( E_{cn} = 0 \). This theorem, however, reveals that in SIMO systems non-minimum phase zero does not always give a contribution on the optimal performance.

The following corollary shows that how the unstable pole closes to non-minimum phase zero can degrade the optimal performance.

**Corollary 4.1** Suppose that the plant \( P(s) \) has single unstable pole \( p \) and common non-minimum phase zeros \( z_i \ (i = 1, \ldots, n_z) \). Then,

\[ E^*_c = 2p \left( \prod_{i=1}^{n_z} \frac{z_i + p}{z_i - p} \right)^2. \]
4.2 Continuous-time Case

We consider two examples to confirm the validity the expression given by Theorem 4.1.

**Example 4.1** Consider an SISO plant given by

\[ P(s) = \frac{2s - 3}{s - p}, \]

in which \( P(s) \) has a non-minimum phase zero at \( s = \frac{3}{2} \) and possibly an unstable pole at \( s = p \). We compute the optimal energy regulation performance by using Matlab toolbox and expression of Theorem 4.1. Fig. 4.2 shows that two computations match well. Particularly, whenever \( P(s) \) is stable then \( E_c^* = 0 \) and if \( p \) approaches the non-minimum phase zero then the performance becomes very large. In general, the optimal performance of a SISO plant with one non-minimum phase zero \( z \) and one unstable pole \( p \) is determined by Corollary 4.1 as

\[ E_c^* = 2p \left( \frac{z + p}{z - p} \right)^2. \]

**Example 4.2** We consider an SISO plant

\[ P(s) = \frac{(4s - 3)(s - z)}{s^2 - 4}, \]

which possesses non-minimum phase zeros at \( s = \frac{3}{4} \) and possibly at \( s = z \) and unstable pole at \( s = 2 \). We compute the optimal energy regulation for \( z \) from \(-1\) to \( 4 \). Fig. 4.3 plots the results.

4.2.2 Output Regulation Problem

Result on the output regulation problem of MIMO system is provided in [14]. We extend the result in two senses: we consider a problem for non-minimum
Fig. 4.3. The energy regulation performance for continuous-time system (Example 4.2).

phase system as well as that under a sensitivity constraint, even our result is only valid for SIMO system.

In order for $E_c$ in (4.9) to be finite, it is necessary that $P(s)d(s) \in \mathcal{L}_2$ and $W_s(s)d(s) \in \mathcal{L}_2$. Since $d(t)$ is an impulse function so that $\hat{d}(s) = 1$, we need the following assumption.

**Assumption 4.1** $P(s)$ and $W_s(s)$ are strictly proper.

We are now ready to provide the best achievable output regulation performance. We give the following expression of (4.13).

**Theorem 4.2** Suppose that the plant $P(s)$ given in (2.4) has common non-minimum phase zeros $z_i (i \in \mathbb{N}_z)$ and unstable poles $p_k (k = 1, \ldots, n_p)$. Define the inner-outer factorization such that

$$
\begin{bmatrix}
W_s(s) \\
W_g(s)N(s) \\
-1
\end{bmatrix} = \Lambda_i(s)\Lambda_o(s).
$$

Then, under Assumption 4.1, the optimal output regulation performance is given by

$$
E_c^* = E_{cm} + E_{cn},
$$

(4.17)
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\[ E_{cm} := 2 \sum_{k=1}^{n_p} p_k + \frac{1}{\pi} \int_{0}^{\infty} \log \left( 1 + \|W_s(j\omega)\|^2 + \|W_y(j\omega)P(j\omega)\|^2 \right) d\omega, \]

\[ E_{cn} := \sum_{i,j \in H_z} 4 \text{Re} \frac{z_i \text{Re} z_j}{\alpha_i \alpha_j (z_i + z_j)} \alpha_i \alpha_j \]

with

\[ \alpha_i := \begin{cases} \prod_{j \in \mathbb{N}_z, j \neq i} \frac{z_j - z_i}{z_j + z_i} ; \# \mathbb{N}_z = 1, \\ 1 ; \# \mathbb{N}_z \geq 2, \end{cases} \]

\[ \alpha_i := 1 - A_o(z_i) \prod_{k=1}^{n_p} \frac{z_i + p_k}{z_i - p_k}. \]

Proof: See Appendix A.3.

Theorem 4.2 shows that if we simultaneously optimize the energy of the control input and the system output as well as the sensitivity reduction, then we have an additional integral term imposed by the gain of the plant and that of the weighting functions. Meanwhile, the plant unstable poles and non-minimum phase zeros contribute their effects in a similar manner as do in the energy regulation problem, except there is an effect caused by \( A_o(s) \). In some cases we may obtain an explicit formula for \( A_o(s) \) if \( W_y = 0 \), since \( A_o(s) \) does not depend on the plant \( P(s) \). In general, since \( A_o \) is stable and minimum phase, its absolute value at \( z_i \) can be obtained from Lemma 4.2, that is

\[ |A_o(z_i)| = \exp \left\{ \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left( \frac{1 + j\omega z_i}{z_i + j\omega} \right) \frac{\log |A_o(j\omega)|^2}{1 + \omega^2} d\omega \right\}, \]

where

\[ |A_o(j\omega)|^2 = 1 + \|W_s(j\omega)\|^2 + \|W_y(j\omega)P(j\omega)\|^2. \]

 Particularly, if \( W_s(s) = 0 \) and \( W_y(s) = 0 \), which imply the integral term is zero and \( A_o(s) = 1 \), then we can recover the results of energy regulation problem obtained in Theorem 4.1.

Remark: The complexity of \( E_{cn} \)-term can be reduced by considering simple cases. For stable case we may obtain

\[ E_{cn} = \sum_{i,j \in H_z} 4 \text{Re} \frac{z_i \text{Re} z_j}{\alpha_i \alpha_j (z_i + z_j)} [1 - A_o(z_i)][1 - A_o(z_j)]. \]

And whenever the plant has single unstable pole \( p \) and single non-minimum phase zero \( z \), we have

\[ E_{cn} = 2z \left[ 1 - A_o(z) \frac{z + p}{z - p} \right]^2, \]
which shows that unstable pole closes to non-minimum phase zero will deteriorate the performance.

Now we pick one example to confirm the effectiveness of the derived expression in Theorem 4.2.

**Example 4.3** Consider an SISO plant described by

\[ P(s) = \frac{s - 3}{(4s - 1)(s - p)}. \]

Clearly, \( P(s) \) has one non-minimum phase zero at \( s = 3 \) and possibly two unstable poles at \( s = \frac{1}{4} \) and \( s = p \). We compute the optimal output regulation performance \( E^*_c \) obtained by Theorem 4.2 (circled-line) and numerically calculated by Matlab toolbox (starred-line) for \( p \) from \(-1\) to \(6\). Here we take the weighting functions as follow:

\[ W_s(s) = \frac{1}{3s + 1}, \quad W_y(s) = \frac{2s + 1}{3s + 1}. \]

Fig. 4.4 shows that two computations match rather well. Particularly, when \( p \) closes to \( 3 \), the performance will be unbounded since it happens almost unstable pole-zero cancellation.

### 4.3 Discrete-time Case

Now we provide the analytical closed-form expression of the optimal regulation performance for discrete-time systems. It is worth to point out that in the regulation problem, we have to exploit a certain function evaluated at infinity, which is laid on the \( j\omega \)-axis (boundary of \( s \)-domain) but not on the unit
4.3 Discrete-time Case

circle (boundary of $z$-domain). It means that derivation process in discrete-time case is not parallel with that of continuous-time case. This is contrast with the tracking problem, where the derivations for the discrete-time are almost parallel to those for the continuous-time case, see the derivation of Theorems 3.1, 3.2, 3.3, and 3.4, 3.5, 3.6.

Recall the coprime factorizations of $P(z)$ in (2.4). Without loss of generality, it is possible to set

$$M(z) = B(z) := \prod_{k=1}^{n_\lambda} \frac{z - \lambda_k}{\bar{\lambda}_k z - 1}$$

(4.18)

where $\lambda_k (k = 1, \ldots, n_\lambda)$ are the unstable poles of $P(z)$. It is important to note that $B(z)$ is inner function and

$$B(\infty) = \prod_{k=1}^{n_\lambda} \frac{1}{\lambda_k}.$$  

To facilitate our derivation we define

$$\hat{N}(z) = z \tilde{N}(z)$$

(4.19)

and introduce the following index set:

$$\mathbb{N}_\eta := \{i : \tilde{N}(\eta_i) = 0, \eta_i \in \mathbb{D}^c\}.$$  

(4.20)

The definition of $\hat{N}(z)$ indicates that we decrease by one the relative degree of $P(z)$ since we may allow the implementation of the biproper controllers. While, $\mathbb{N}_\eta$ contains the index set of all common non-minimum phase zeros of $P(z)$ with counting multiplicities except one zero at infinity, i.e., the number of zeros at infinity in $\mathbb{N}_\eta$ is equal to the relative degree of $P(z)$ minus one.

4.3.1 Energy Regulation Problem

Here we provide the minimal value of (4.7), i.e.,

$$E_d^* = \inf_{K \in \mathcal{K}} \sum_{k=0}^{\infty} |u(k)|^2.$$  

In other words, we derive the analytical closed-form expression of the optimal energy regulation performance. Note that the optimal performance is determine by (4.8).

**Theorem 4.3** Suppose that the SIMO plant $P(z)$ is given in (2.4) has common non-minimum phase zeros $\eta_i (i \in \mathbb{N}_\eta)$ and unstable poles $\lambda_k (k = 1, \ldots, n_\lambda)$. Then the optimal energy regulation performance is given by
where
\[
E_{dm} := \prod_{k=1}^{n_{\lambda}} |\lambda_k|^2 - 1,
\]
\[
E_{dn} := \sum_{i,j \in \mathbb{N}_{\eta}} \frac{(|\eta_i|^2 - 1)(|\eta_j|^2 - 1)}{b_i b_j (\eta_i \eta_j - 1)} \hat{\beta}_i \beta_j
\]
with
\[
b_i := \begin{cases} 
1 & ; \# \mathbb{N}_{\eta} = 1 \\
\prod_{j \in \mathbb{N}_{\eta}, j \neq i} \frac{\eta_j - \eta_i}{\eta_j \eta_i - 1} & ; \# \mathbb{N}_{\eta} \geq 2
\end{cases}
\]
\[
\beta_i := \prod_{k=1}^{n_{\lambda}} \lambda_i - \prod_{k=1}^{n_{\lambda}} \frac{\lambda_i \eta_i - 1}{\eta_i - \lambda_k}.
\]

Proof. See Appendix A.4. □

The most important fact revealed by Theorem 4.3 is that the contribution of unstable poles is given in product way instead of in summation like in continuous-time case, see Theorem 4.1. We may explain this difference as follows. Suppose that the continuous-time plant \(P(s)\) has unstable poles \(p_k (k = 1, \ldots, n_p)\). Then the discretization process with sampling time \(T\) seconds will produce a discrete-time plant \(P(z)\), which has unstable poles \(\lambda_k (k = 1, \ldots, n_{\lambda})\), where \(\lambda_k = e^{p_k T}\). It is obvious that
\[
\prod_{k=1}^{n_{\lambda}} \lambda_k = \exp \left\{ T \sum_{k=1}^{n_p} p_k \right\}.
\]
This, however, gives an insight that in discrete-time case unstable poles contribute more detrimental effect than those in continuous-time case. For SIMO case, condition \(\eta_i (i \in \mathbb{N}_{\eta})\) means that only the common non-minimum phase zeros give effect. Hence, if \(P(z)\) has no common non-minimum phase zero then \(E_{dn} = 0\).

Example 4.4 We consider the following SISO plant
\[
P(z) = \frac{1}{z - \lambda},
\]
in which \(P(z)\) has relative degree 1 and possibly an unstable pole at \(z = \lambda\). Obviously, if \(|\lambda| \leq 1\) then \(E^*_d = 0\). For \(|\lambda| > 1\) we consider two cases. First, we implement a biproper controller. The optimal performance is then given by
\[
E^*_d = \lambda^2 - 1.
\]
Second, we apply a strictly proper controller. By this assumption, plant \( P(z) \) has one non-minimum phase zero at \( z = \infty \). Hence,

\[
E_a^* = (\lambda^2 - 1) + \lim_{\eta \to \infty} (\lambda^2 - 1)^2 \frac{\eta^2 - 1}{(\eta - \lambda)^2} = (\lambda^2 - 1)\lambda^2.
\]

We also numerically compute the optimal performance by using Matlab toolbox. Fig. 4.5 depicts the computation results for \( \lambda \) from \(-2 \) to \( 2 \).

Now we provide two direct implications of Theorem 4.3 in the following corollaries.

**Corollary 4.2** Suppose that the SIMO plant \( P(z) \) given in (2.4) is minimum phase, has relative degree 2 and unstable poles \( \lambda_k \) \((k = 1, \ldots, n_{\lambda})\). Then,

\[
E_a^* = E_{dm} + E_{dn}, \tag{4.22}
\]

where

\[
E_{dm} := \prod_{k=1}^{n_{\lambda}} |\lambda_k|^2 - 1,
\]

\[
E_{dn} := \left[ \prod_{k=1}^{n_{\lambda}} |\lambda_k|^2 \right] \left[ \sum_{k=1}^{n_{\lambda}} \frac{|\lambda_k|^2 - 1}{\lambda_k} \right]^2.
\]

**Proof.** The proof of \( E_{dm} \) is obvious. If \( P(z) \) has only one unstable \( \lambda \) then from the expression in Theorem 4.3 we obtain

\[
E_{dn} = (\lambda^2 - 1)^2 = \lambda^2 \left( \lambda - \frac{1}{\lambda} \right)^2 = |\lambda|^2 \left| \frac{|\lambda|^2 - 1}{\lambda} \right|^2.
\]

If \( P(z) \) has two unstable poles \( \lambda_1, \lambda_2 \) then
\[ E_{dn} = (\lambda_1^2 + \lambda_2 - \lambda_1 - \lambda_2)^2 \]

Subsequently, if \( P(z) \) has three unstable poles \( \lambda_1, \lambda_2, \lambda_3 \) then

\[ E_{dn} = |\lambda_1|^2|\lambda_2|^2|\lambda_3|^2 \left( \frac{|\lambda_1|^2 - 1}{\lambda_1} + \frac{|\lambda_2|^2 - 1}{\lambda_2} + \frac{|\lambda_3|^2 - 1}{\lambda_3} \right)^2. \]

In general, if \( P(z) \) has \( n_\lambda \) unstable poles \( \lambda_k \) (\( k = 1, \ldots, n_\lambda \)) then

\[ E_{dn} = \left[ \prod_{k=1}^{n_\lambda} |\lambda_k|^2 \right] \left[ \sum_{k=1}^{n_\lambda} \left| \frac{|\lambda_k|^2 - 1}{\lambda_k} \right| \right]^2. \]

It is proved. Note that the expressions in this corollary are similar to those in Proposition 4.3 of [42].

**Corollary 4.3** Suppose that the SIMO plant \( P(z) \) given in (2.4) has relative degree \( v \), common non-minimum phase zeros \( \eta_i \) (\( i \in \mathbb{N}_\eta \)), and only one unstable pole \( \lambda \). Then,

\[ E_d^* = \lambda^2(\lambda^2 - 1) \left( \prod_{i \in \mathbb{N}_\eta} \frac{\lambda \eta_i - 1}{\eta_i - \lambda} \right)^2. \]

*Proof.* Let the plant \( P(z) \) has only one unstable pole \( \lambda \). In addition, if \( P(z) \) has relative degree 1 and one common non-minimum phase zero \( \eta \), then from the expressions in Theorem 4.3 we obtain

\[ E_d^* = (\lambda^2 - 1) \left( \frac{\lambda \eta - 1}{\eta - \lambda} \right)^2. \]

If \( P(z) \) has relative degree 2 and two common non-minimum phase zeros \( \eta_1, \eta_2 \), then

\[ E_d^* = \lambda^2(\lambda^2 - 1) \left( \frac{\lambda \eta_1 - 1}{\eta_1 - \lambda} \frac{\lambda \eta_2 - 1}{\eta_2 - \lambda} \right)^2. \]

Furthermore, if \( P(z) \) has relative degree 3 and three common non-minimum phase zeros \( \eta_1, \eta_2, \eta_3 \), then

\[ E_d^* = \lambda^4(\lambda^2 - 1) \left( \frac{\lambda \eta_1 - 1}{\eta_1 - \lambda} \frac{\lambda \eta_2 - 1}{\eta_2 - \lambda} \frac{\lambda \eta_3 - 1}{\eta_3 - \lambda} \right)^2. \]

In general, if \( P(z) \) has relative degree \( v \) and common non-minimum phase zeros \( \eta_i \) (\( i \in \mathbb{N}_\eta \)), then

\[ E_d^* = \lambda^{2(v-1)}(\lambda^2 - 1) \left( \prod_{i \in \mathbb{N}_\eta} \frac{\lambda \eta_i - 1}{\eta_i - \lambda} \right)^2. \]

It is proved. ■
4.3.2 Output Regulation Problem

Here we provide the minimal value of (4.10), i.e.,

$$E^*_d = \inf_{K \in \mathbb{K}} \sum_{k=0}^{\infty} \left( \|y_w(k)\|^2 + |u(k)|^2 \right),$$

which further expressed by (4.13). Note that the plant $P(z)$ and the weighting function $W_s(z)$ are not necessary to be strictly proper to regulate a discrete-time system.

**Theorem 4.4** Suppose that the SIMO plant $P(z)$ is given in (2.4) has common non-minimum phase zeros $\eta_i$ ($i \in \mathbb{N}_\eta$) and unstable poles $\lambda_k$ ($k = 1, \ldots, n_\lambda$). Define the inner-outer factorization such that

$$\begin{bmatrix} W_s(z) \\ W_y(z)N(z) \\ -1 \end{bmatrix} = A_i(z)A_o(z).$$

Then the optimal output regulation performance is given by

$$E^*_d = E_{dm} + E_{dn}, \quad (4.24)$$

where

$$E_{dm} := \exp \left\{ \frac{1}{\pi} \int_0^{\pi} \log \left( 1 + \|W_s(e^{j\theta})\|^2 + \|W_y(e^{j\theta})P(e^{j\theta})\|^2 \right) \, d\theta \right\} \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1,$$

$$E_{dn} := \sum_{i,j \in \mathbb{N}_\eta} \frac{(|\eta_i|^2 - 1)(|\eta_j|^2 - 1)}{b_i b_j (\eta_i \eta_j - 1)} \beta_i \beta_j$$

with

$$b_i := \begin{cases} 1 & ; \#\mathbb{N}_\eta = 1 \\ \frac{\eta_j - \eta_i}{\eta_i \eta_j - 1} & ; \#\mathbb{N}_\eta \geq 2, \end{cases}$$

$$\beta_i := A_o(\infty) \prod_{k=1}^{n_\lambda} \lambda_k - A_o(\eta_i) \prod_{k=1}^{n_\lambda} \frac{\lambda_k \eta_i - 1}{\eta_i - \lambda_k}.$$

**Proof.** See Appendix A.5. \qed

Theorem 4.4 points-out that when we minimized the energy of control input simultaneously with that of system output then an additional term caused by the plant gain influences the optimal performance. Differ from its continuous-time counterpart in Theorem 4.2, the effect of the plant gain is expresses in exponential way rather than in linear manner. This fact suggests that the plant gain contributes more detrimental effects than those in
continuous-time case. The remainders are almost similar with those of the energy regulation problem, i.e., Theorem 4.3, except the existence of the outer function \( \Lambda_o(z) \). The expression for \( \Lambda_o(\infty) \) and \( |\Lambda_o(\eta_i)| \) can be derived from Poisson-Jensen formula in Lemma 4.1 as follow:

\[
\Lambda_o(\infty) = \exp \left\{ \frac{1}{2\pi} \int_0^{\pi} \log |A_o(e^{i\theta})|^2 \, d\theta \right\},
\]

\[
|\Lambda_o(\eta_i)| = \exp \left\{ \frac{1}{2\pi} \int_0^{\pi} \Re \left( \frac{\eta_i e^{i\theta} + 1}{\eta_i e^{i\theta} - 1} \right) \log |A_o(e^{i\theta})|^2 \, d\theta \right\},
\]

where

\[
|A_o(e^{i\theta})|^2 = 1 + \|W_s(e^{i\theta})\|^2 + \|W_y(e^{i\theta})P(e^{i\theta})\|^2.
\]

Again, if \( W_s(z) = 0 \) and \( W_y(z) = 0 \) then the expression reduces to that of Theorem 4.3. We can also consider some special cases relate to the type of the plant. Let introduce the following mild definition. The plant \( P(z) \) is called non-minimum phase if \( P(z) \) has common non-minimum phase zero. Otherwise, minimum phase.

**Corollary 4.4** Suppose that the SIMO plant \( P(z) \) is given in (2.4). The followings are direct implications of Theorem 4.4:

1. If \( P(z) \) is unstable and minimum phase (possibly with relative degree 1), then

\[
E^*_d = \exp \left\{ \frac{1}{\pi} \int_0^{\pi} \log \left( 1 + \|W_s(e^{i\theta})\|^2 + \|W_y(e^{i\theta})P(e^{i\theta})\|^2 \right) \, d\theta \right\} \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1.
\]

2. If \( P(z) \) is stable and non-minimum phase then \( E^*_d = E_{\text{dm}} + E_{\text{dn}} \), where

\[
E_{\text{dm}} := \exp \left\{ \frac{1}{\pi} \int_0^{\pi} \log \left( 1 + \|W_s(e^{i\theta})\|^2 + \|W_y(e^{i\theta})P(e^{i\theta})\|^2 \right) \, d\theta \right\} - 1,
\]

\[
E_{\text{dn}} := \sum_{i,j \in N_o} \frac{(|\eta_i|^2 - 1)(|\eta_j|^2 - 1)}{b_i b_j (\eta_i \eta_j - 1)} (A_o(\infty) - A_o(\eta_i))(A_o(\infty) - A_o(\eta_j)).
\]

3. If \( P(z) \) is stable and minimum phase then

\[
E^*_d = \exp \left\{ \frac{1}{\pi} \int_0^{\pi} \log \left( 1 + \|W_s(e^{i\theta})\|^2 + \|W_y(e^{i\theta})P(e^{i\theta})\|^2 \right) \, d\theta \right\} - 1.
\]

The following illustrative example confirms the validity of the result in Theorem 4.4.

**Example 4.5** We consider an SISO plant given by

\[
P(z) = \frac{4z^2 - 9}{z(3z + 4)(z - \lambda)},
\]
which has non-minimum phase zeros at \( z = \frac{3}{2} \) and \( z = -\frac{3}{2} \), and unstable poles at \( z = -\frac{4}{3} \) and possibly at \( z = \lambda \). Fig. 4.6 plots Theorem 4.4 based computation (circled-line) and MATLAB toolbox-based computation (starred-line) for \( \lambda \) from \(-3\) to 3, where we set \( W_s(z) = 1 \) and \( W_y(z) = 1 \). The figure clearly shows that the expression in Theorem 4.4 is correct and that \( E_d^* \) becomes larger when \( \lambda \) approaches to one of the non-minimum phase zeros.

### 4.3.3 Output Regulation Problem with Noise

In the recent years, there is a growing attention related to the research activity in feedback control with communication constraints [4, 42]. In this part, we consider a feedback control system in which there exists a communication link, see Fig. 4.7. In digital communication, the link consists of some pre and post processing equipments for the signal that are sent through the communication channel, which might be in the form of filter, A-D converter, coder, modulator, decoder, demodulator, and D-A converter.

In this study, an signal-to-noise ratio constrained channel will be considered and all pre and post signal processing are restricted to LTI filtering and D-A/A-D type operations. Thus, the communication link simplifies to the

![Diagram of feedback control system with communication link](image-url)
noisy channel, Fig. 4.8. Here, \( y \in \mathbb{R}^m \) is the measurement output and \( n \in \mathbb{R}^m \) is a zero mean additive white Gaussian noise with intensity \( \Omega \), i.e.,

\[
\mathcal{E}[n(k)] = 0, \quad \mathcal{E}[n(k)n^T(\kappa)] = \Omega \delta(k - \kappa),
\]

where \( \mathcal{E}[\cdot] \) represents the expectation operator and \( \delta \) is the unitary impulse function:

\[
\delta(k - \kappa) = \begin{cases} 
1, & k = \kappa \\
0, & k \neq \kappa.
\end{cases}
\]

We define

\[
\|y\| = \sqrt{\mathcal{E}[y'y]},
\]

and assume a given energy constraint \( \mathcal{Y} \), such that it is required

\[
\|y\|^2 < \mathcal{Y},
\]

for some predetermined value \( \mathcal{Y} > 0 \). The stabilization problem addressed in this section is then can be stated as a problem of finding the smallest value of \( \|y\| \). Further we may write

\[
\|y\|^2 = \|T(z)\|^2 \Omega, \quad (4.25)
\]

where \( T = PK(I + PK)^{-1} \). Note that by taking \( \Omega \) as an identity, we then have a the problem of stabilization which minimize

\[
E_d := \sum_{k=0}^{\infty} \|y(k)\|^2. \quad (4.26)
\]

By Parseval identity we can write

\[
E_d = \|T(z)\hat{n}(z)\|^2 = \|T(z)1\|^2_2 = \|(I - S_o(z))1\|^2_2,
\]

where \( S_o(z) \) is the sensitivity function defined in (3.2). Then we can write the optimal performance as

\[
E_d^* = \inf_{Q \in \mathcal{KH}_\infty} \|I - (X - NQ)\hat{M}|1\|^2_2. \quad (4.27)
\]

Without loss of generality we may set \( \hat{M}(z) = B(z)I \), where \( B(z) \) is given by (4.18). Hence,
4.4 Delta Domain Case

\[ E_d^* = \inf_{Q \in \mathcal{RH}_\infty} \| (B^{-1}I - X + NQ)1 \|_2^2 \]
\[ = \inf_{Q \in \mathcal{RH}_\infty} \| ((B^{-1} - B^{-1}(\infty))I + (B^{-1}(\infty)I - X + NQ))1 \|_2^2 \]
\[ = \| (B^{-1} - B^{-1}(\infty))I \|_2^2 + \inf_{Q \in \mathcal{RH}_\infty} \| (B^{-1}(\infty)I - X + NQ)1 \|_2^2. \]

Now we are ready to provide the analytical closed-form expression of \( E_d^* \).

**Theorem 4.5** Consider the feedback control setup given by Fig. 4.8, where \( P(z) \) is SIMO plant given in (2.3), which has common non-minimum phase zeros \( \eta_i (i \in \mathbb{N}_\eta) \) and unstable poles \( \lambda_k (k = 1, \ldots, n_\lambda) \). Then the optimal output regulation performance is given by

\[ E_d^* = E_{dm} + E_{dn}, \quad (4.28) \]

where

\[ E_{dm} := m \left[ \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1 \right], \]
\[ E_{dn} := m \sum_{i,j \in \mathbb{N}_\eta} (|\eta_i|^2 - 1)(|\eta_j|^2 - 1) \frac{\beta_i \beta_j}{b_i b_j (\eta_i \eta_j - 1)} \]

with

\[ b_i := \begin{cases} 1 & ; \#\mathbb{N}_\eta = 1 \\ \prod_{j \in \mathbb{N}_\eta, j \neq i} \frac{\eta_i - \eta_j}{\eta_j \eta_i - 1} & ; \#\mathbb{N}_\eta \geq 2 \end{cases}, \]
\[ \beta_i := \prod_{k=1}^{n_\lambda} \lambda_k - \prod_{k=1}^{n_\lambda} \frac{\lambda_k \eta_i - 1}{\eta_i - \lambda_k}. \]

**Proof.** Follow the proof of Theorem 4.3. \[ \blacksquare \]

Note that if \( m = 1 \), i.e., we consider an SISO plant \( P(z) \), then the expressions of this theorem is similar with those of Theorem 4.3. It can be understood since \( \| y \|_2^2 \) is completely determined by the complementary sensitivity function \( T(z) \), then the problem can be similarly treated as an energy regulation problem.

### 4.4 Delta Domain Case

In this section we reformulate and solve the optimal regulation problem in term of the delta operator in order to link the continuous-time and discrete-time results derived in the previous sections.

Consider the coprime factorizations of \( P(\delta) \) given in (2.4), in which without loss of generality we may set \( M = H \), where
\[ H(\delta) := B(T\delta + 1) = \prod_{k=1}^{n_\lambda} \frac{(T\delta + 1) - \lambda_k}{\lambda_k(T\delta + 1) - 1}, \]  
(4.29)

with \( \lambda_k \) are the unstable poles of \( P(z) \). It is easy to show that \( H \) is inner in delta domain, i.e., \( H(e^{-j\omega T} - 1)H(e^{j\omega T} - 1) = 1 \) and \( H(\infty) = B(\infty) \). We remark that \( H \) possesses non-minimum phase zeros \( \rho_k \in \overline{D}_cT \) at \( \rho_k = (\lambda_k - 1)/T \) for \( k = 1, 2, \ldots, n_\rho \), in which they also act as the unstable poles of \( P(\delta) \). Note that \( n_\rho = n_\lambda \).

As the disturbance input signal, we consider an impulse function in the following form:

\[
d(k) = \begin{cases} 
\frac{1}{T}, & \text{for } k = 0 \\
0, & \text{for } k \neq 0 
\end{cases},
\]
(4.30)

where its \( \mathcal{D} \)-transform is \( \hat{d}_T(\delta) = 1 \). We define

\[
\tilde{N}(\delta) = \delta \tilde{N}(\delta),
\]
(4.31)

and introduce the following index set:

\[
\mathbb{N}_\zeta := \{ i : \tilde{N}(\zeta_i) = 0, \zeta_i \in \overline{D}_cT \}.
\]
(4.32)

Note that \( \mathbb{N}_\zeta \) contains the index set of all common non-minimum phase zeros of \( P(\delta) \) with counting multiplicities except one zero at infinity.

### 4.4.1 Energy Regulation Problem

Here we reformulate and solve the energy regulation problem, which is previously discussed in Subsection 4.3.1, in terms of delta operator. In other words, we minimize the following performance index

\[
E_\delta := T \sum_{k=0}^{\infty} |u(k)|^2.
\]
(4.33)

By Parseval’s identity we may write (4.33) as

\[
E_\delta = \| K(\delta)S_u(\delta)P(\delta)\hat{d}_T(\delta) \|_2^2.
\]

The optimal performance is then deduced as

\[
E_\delta^* = \inf_{Q \in \mathbb{R}H_\infty} \| Y \tilde{N} - MQ\tilde{N} \|_2^2,
\]
(4.34)

The analytical closed-form expression of the optimal performance (4.34) is provided in the following theorem.
Theorem 4.6 Suppose that the plant $P(\delta)$ given in (2.4) has common non-minimum phase zeros $\zeta_i (i \in N_\zeta)$ and unstable poles $\rho_k (k = 1, \ldots, n_\rho)$. Then the optimal energy regulation performance is given by

$$E^*_\delta = E_{\delta m} + E_{\delta n},$$

where

$$E_{\delta m} := \frac{1}{T} \prod_{k=1}^{n_\rho} |T\rho_k + 1|^2 - 1,$$

$$E_{\delta n} := \sum_{i,j \in N_\zeta} \frac{(T|\zeta_i|^2 + 2 \text{Re}\, \zeta_i)(T|\zeta_j|^2 + 2 \text{Re}\, \zeta_j)}{g_i g_j (T\zeta_i + \zeta_j + \zeta_j)} \gamma_i \gamma_j$$

with

$$g_i := \begin{cases} 1 & \text{if } \#N_\zeta = 1 \\ \prod_{j \in N_\zeta, j \neq i} \frac{\zeta_i - \zeta_j}{T\zeta_i \zeta_j + \zeta_j + \zeta_i} & \text{if } \#N_\zeta \geq 2 \end{cases}$$

$$\gamma_i := \prod_{k=1}^{n_\rho} (T\rho_k + 1) - \prod_{k=1}^{n_\rho} \frac{T\rho_k \zeta_i + \bar{\rho}_k + \zeta_i}{\zeta_i - \rho_k}.$$

Proof. By invoking the proof of Theorem 4.3 we can immediately write

$$E^*_\delta = E_1 + E_2,$$

where

$$E_1 := \|H^{-1}(\infty) - H^{-1}(\delta)\|_2^2,$$

$$E_2 := \inf_{Q \in \mathcal{R} H_\infty} \|H^{-1}(\infty) - \tilde{X} + Q\tilde{N}\|_2^2.$$

Note that since $H$ is inner,

$$E_1 = \|H^{-1}(\infty)H(\delta) - 1\|_2^2 = \frac{1}{T}\|B^{-1}(\infty)B(z) - 1\|_2^2.$$

The last holds from the norm relation (2.20). We then show that $E_1 = E_{\delta m}$ by fact that

$$\|B^{-1}(\infty)B(z) - 1\|_2^2 = \prod_{k=1}^n |\lambda_k|^2 - 1 = \prod_{k=1}^{n_\rho} |T\rho_k + 1|^2 - 1.$$

We can show that $E_2 = E_{\delta n}$ by performing the partial fraction expansion as did in the proof of Theorem 4.3. The factor $\frac{1}{T}$ appears from the fact that

$$\left\| \frac{1}{(T\delta + 1) - \eta_k} \right\|_2^2 = \frac{1}{T} \left\| \frac{1}{z - \eta_k} \right\|_2^2 = \frac{1}{T} \frac{1}{\eta_k^2 - 1},$$

where $\eta_k = T\zeta_i + 1$. We complete the proof. ■
4.4.2 Output Regulation Problem

We consider the following performance index

\[ E_\delta := T \sum_{k=0}^{\infty} (\|y_w(k)\|^2 + |u(k)|^2), \tag{4.36} \]

where \( y_w(k) \) is given by

\[ y_w(k) = \begin{bmatrix} y_{w1}(k) \\ y_{w2}(k) \end{bmatrix} = \begin{bmatrix} D^{-1} \{ W_s(\delta)[\hat{u}(\delta) + \hat{d}(\delta)] \} \\ D^{-1} \{ W_y(\delta)\hat{y}(\delta) \} \end{bmatrix}. \]

The optimal output regulation performance is also provided by (4.13).

Now we are ready to provide the analytical expressions of the optimal performance in delta domain. First we reformulate Lemmas 4.1 and 4.5 in delta domain, respectively as follow.

**Lemma 4.6** Let \( h \) be analytic in \( \bar{D}_c \) and \( \sigma_i (i = 1, \ldots, n) \) be the zeros of \( h \) in \( \bar{D}_c \), counting their multiplicities. If \( \delta \in \bar{D}_c \) and \( h(\delta) \neq 0 \), then

\[ \log |h(\delta)| = \frac{T}{\pi} \int_0^{\pi/T} \text{Re} \left( \frac{(T\delta + 1)e^{j\omega T} + 1}{(T\delta + 1)e^{j\omega T} - 1} \right) \log \left| \frac{e^{j\omega T} - 1}{T} \right| d\omega - \sum_{i=1}^{n} \log \left| \frac{T\delta - \sigma_i + \delta}{\delta - \sigma_i} \right|. \tag{4.37} \]

**Lemma 4.7** If \( h \in \mathbb{R}H_\infty \), then

\[ \frac{T}{\pi} \int_0^{\pi/T} \text{Re} h \left( \frac{e^{j\omega T} - 1}{T} \right) d\omega = h(\infty). \tag{4.38} \]

**Theorem 4.7** Suppose that the plant \( P(\delta) \) given in (2.4) has common non-minimum phase zeros \( \zeta_i (i \in \mathbb{N}_\zeta) \) and unstable poles \( \rho_k (k = 1, \ldots, n_p) \). Define the inner-outer factorization such that

\[ \begin{bmatrix} W_s(\delta) \\ W_y(\delta)N(\delta) \\ -1 \end{bmatrix} = A_i(\delta)A_o(\delta). \]

Then the optimal output regulation performance is given by

\[ E^*_\delta = E_{\delta m} + E_{\delta n}, \tag{4.39} \]

where

\[ E_{\delta m} := \frac{1}{T} \left| A_o(\infty) \right|^2 \prod_{k=1}^{n_p} |T\rho_k + 1|^2 - 1, \]

\[ E_{\delta n} := \sum_{i,j \in \mathbb{N}_\zeta} \frac{(T|\zeta_i|^2 + 2 \text{Re} \zeta_i)(T|\zeta_j|^2 + 2 \text{Re} \zeta_j)\hat{\varphi}_{i,j}}{g_i g_j(T\zeta_i \zeta_j + \zeta_i + \zeta_j)} \hat{\gamma}_{i,j}. \]
4.4 Delta Domain Case

$$|\Lambda_0(\infty)|^2 = \exp\left\{ \frac{T}{\pi} \int_0^{\pi/T} \log(1 + \|W_s(e^{j\omega T} - 1)/T\|^2 + \|W_s(e^{j\omega T} - 1)P(e^{j\omega T} - 1)\|^2) \, d\omega \right\},$$

$$g_i := \begin{cases} 1 - \prod_{j \in \mathbb{N}_z, j \neq i} \frac{z_i - z_j}{Tz_j + z_i} & \text{if } \#\mathbb{N}_z = 1, \\
\prod_{j \in \mathbb{N}_z, j \neq i} \frac{z_i - z_j}{Tz_j + z_i} & \text{if } \#\mathbb{N}_z \geq 2, \end{cases}$$

$$\gamma_i := A_0(\infty) \prod_{k=1}^{\eta_i} (T\rho_k + 1) - A_0(\zeta_j) \prod_{k=1}^{\eta_i} \frac{T\rho_k z_i + \rho_k + z_i}{z_i - \rho_k}.$$

**Proof.** The proof is almost parallel to the discrete-time case, i.e., the proof of Theorem 4.4. Use Lemma 4.7. □

Theorem 4.7 shows that the expressions of the optimal performance are not only explicitly characterized by the unstable poles and non-minimum phase zeros of the plant but also by the sampling time. Actually, it is not difficult to deduce that $E_\delta^* = E_\delta^*/T$, where we impose the relations $\lambda_k = T\rho + 1$ and $\eta_i = T\zeta_i + 1$ and note that the expression of $|\Lambda_0(\zeta_i)|$ can be derived by Lemma 4.6.

In the next section we will inspect the convergence of the expressions of Theorem 4.7 when we let the sampling time approach zero.

4.4.3 Continuity Property

Now we will demonstrate that $E_\delta^*$ converges to $E_c^*$ when the sampling time $T$ tends to zero. We deal with the output regulation case. We follow the way of the tracking performance case in Subsection 3.4.4.

We denote by

$$P_c(s) = (P_{c1}(s), P_{c2}(s), \ldots, P_{cn}(s))^T,$$

the respecting continuous-time plant. Suppose that $P_c(s)$ has unstable poles $p_k (k = 1, \ldots, n_p)$ and common non-minimum phase zeros $z_i (i \in \mathbb{N}_z)$. Under the zero-order hold operations with sufficiently small sampling time we obtain the corresponding delta domain plant

$$P_T(\delta) = (P_{T1}(\delta), P_{T2}(\delta), \ldots, P_{Tm}(\delta))^T,$$

where $P_T(\delta)$ has unstable poles $\rho_k (k = 1, \ldots, n_p)$ with $n_p = n_{r_p}$ and common non-minimum phase zeros $\zeta_i (i \in \mathbb{N}_z)$ and $\zeta_*^i (i \in \mathbb{N}_z^*)$. Note that the latter are the limiting zeros of $P_T(\delta)$.

Recall the following pole-zero relationships:
\[
\rho_k = \frac{e^{\mu_k T} - 1}{T}, \\
\zeta_i = \frac{e^{z_i T} - 1}{T}, \\
\zeta_i^* = \frac{\eta_i - 1}{T},
\]

where \(\eta_i^*\) are the zeros of polynomial \(B_r(z)\) defined in (3.66). It is easy to verify that \(E_{\delta m}\) can be written as
\[
E_{\delta m} = E_H + E_R,
\]
where
\[
E_H := \frac{1}{T} \left[ \prod_{k=1}^{n_p} |T\rho_k + 1|^2 - 1 \right],
\]
\[
E_R := \frac{|A_o(\infty)|^2 - 1}{T} \prod_{k=1}^{n_p} |T\rho_k + 1|^2.
\]

Since
\[
E_H \approx 2 \sum_{k=1}^{n_p} \rho_k
\]
and \(\rho_k \to p_k\) as \(T\) tends to zero, then we get
\[
\lim_{T \to 0} E_H = 2 \sum_{k=1}^{n_p} p_k.
\]

Next, since \(e^{j\omega T - 1} \to j\omega\) and \(|T\rho_k + 1|^2 \to 1\) as \(T\) tends to zero, and also \(W_T(\delta), W_T(\delta), P_T(\delta)\) converge to the continuous-time counterparts \(W_{cs}(s), W_{cy}(s), P_c(s)\), we have
\[
\lim_{T \to 0} E_R = \frac{1}{\pi} \int_0^{\infty} \log \left( 1 + ||W_{cs}(j\omega)||^2 + ||W_{cy}(j\omega)P_c(j\omega)||^2 \right) d\omega.
\]

These two facts then show that
\[
\lim_{T \to 0} E_{\delta m} = E_{cm}. \tag{4.40}
\]

We show the convergence of \(E_{\delta m}\) part by part. First we only consider a case where the delta domain plant has no limiting zero. Since \(\zeta_i \to z_i\) as \(T \to 0\), we have
\[
\lim_{T \to 0} g_i = \prod_{j \in \mathbb{N}, j \neq i} \frac{z_j - z_i}{z_j + \bar{z}_i} =: a_i.
\]

Subsequently we obtain
\[
\lim_{T \to 0} \frac{(T|\zeta_i|^2 + 2 \text{Re } \zeta_i)(T|\zeta_j|^2 + 2 \text{Re } \zeta_j)}{g_i g_j (T\zeta_i \zeta_j + \zeta_i + \zeta_j)} = \frac{4 \text{Re } z_i \text{Re } z_j}{a_i a_j (z_i + z_j)^4}.
\]
To inspect the convergence of \( \gamma_i \), we know that \( \lim_{T \rightarrow 0} |A_o(\infty)|^2 = 1 \). Thus, \( \lim_{T \rightarrow 0} A_o(\infty) = 1 \). We also know that

\[
\lim_{T \rightarrow 0} \prod_{k=1}^{n_{\rho}} (T\tilde{\rho}_k + 1) = 1
\]

and

\[
\lim_{T \rightarrow 0} \prod_{k=1}^{n_{\rho}} \frac{T\tilde{\rho}_k \zeta_i + \tilde{\rho}_k + \zeta_i}{\zeta_i - \rho_k} = \prod_{k=1}^{n_{\rho}} \frac{z_i + \tilde{\rho}_k}{z_i - \rho_k}
\]

Now we only need to show the convergence of \( A_{T_o}(\zeta_i) \). From Lemma 4.6 we have

\[
|A_{T_o}(\zeta_i)| = \exp \left\{ \frac{T}{2\pi} \int_0^\pi \text{Re} \left[ \frac{(T\zeta_i + 1)e^{j\omega T} + 1}{(T\zeta_i + 1)e^{j\omega T} - 1} \right] \log \left| \frac{e^{j\omega T} - 1}{T} \right|^2 d\omega \right\}.
\]

Since

\[
\lim_{T \rightarrow 0} T \text{Re} \left[ \frac{(T\zeta_i + 1)e^{j\omega T} + 1}{(T\zeta_i + 1)e^{j\omega T} - 1} \right] = 2 \text{Re} \left[ \frac{1}{z_i + j\omega} \right] = \frac{2}{1 + \omega^2} \text{Re} \left[ \frac{1 + j\omega z_i}{z_i + j\omega} \right]
\]

implies \( \lim_{T \rightarrow 0} A_{T_o}(\zeta_i) = A_{c_o}(z_i) \), then we achieve \( \lim_{T \rightarrow 0} \gamma_i = \alpha_i \). Communicating all the above facts yields

\[
\lim_{T \rightarrow 0} E_{\delta n} = E_{c_n}.
\]

Therefore, (4.40) and (4.41) conclude

\[
\lim_{T \rightarrow 0} E_{\delta}^* = E_{c}^*,
\]

i.e., the \( \delta \)-domain solution converges to the corresponding \( s \)-domain solution as sampling time \( T \) approaches to zero.

Showing the convergence of \( E_{\delta n} \) where there exist some \( i, j \in \mathbb{N}^*_\infty \) is more complicated. Thus, for simplicity we only consider three simplest cases.

**Case 1:** The plant \( P_T(\delta) \) has only one common non-minimum phase zero \( \zeta^* \), i.e., the only zero is a limiting zero. The continuous-time counterpart of this case is that the plant \( P_c(s) \) is minimum phase with relative degree 2, 3, or 4. In this case we have

\[
E_{\delta n} := (T\zeta_i^2 + 2\zeta_i^*)\gamma_i^2,
\]

where

\[
\gamma := A_{T_o}(\infty) \prod_{k=1}^{n_{\rho}} (T\tilde{\rho}_k + 1) - A_{T_o}(\zeta^*) \prod_{k=1}^{n_{\rho}} \frac{T\tilde{\rho}_k \zeta^* + \tilde{\rho}_k + \zeta^*}{\zeta^* - \rho_k}.
\]

Then the following holds:
\[
\lim_{T \to 0} E_{\delta n} = \lim_{\zeta \to \infty} (\eta^* + 1)\zeta^* \left[ A_{\text{co}}(\infty) - A_{\text{co}}(\zeta^*) \prod_{k=1}^{n_p} \frac{\zeta^* + \bar{p}_k}{\zeta^* - p_k} \right]^2 = 0,
\]

which means that the limiting zero \(\zeta^*\) does not give any contribution. Hence, we maintain (4.41).

**Case 2:** The plant \(P_T(\delta)\) has one common non-minimum phase ‘usual’ zero \(\zeta_1 := \zeta\) and one common non-minimum phase limiting zero \(\zeta_2 := \zeta^*\). The continuous-time counterpart of this case is that the plant \(P_c(s)\) has only one minimum phase \(z\) with relative degree 2, 3, or 4. In this case we may write \(E_{\delta n} = E_{\delta n}^1 + E_{\delta n}^2 + E_{\delta n}^3\), where

\[
E_{\delta n}^1 := \frac{(T\zeta_1^2 + 2\zeta_1)\gamma_1^2}{g_1^1},
E_{\delta n}^2 := \frac{(T\zeta_2^2 + 2\zeta_2)\gamma_2^2}{g_1^2},
E_{\delta n}^3 := \frac{2(T\zeta_1^2 + 2\zeta_1)(T\zeta_2^2 + 2\zeta_2)\gamma_1\gamma_2}{g_1^2 g_2^2 (T\zeta_1\zeta_2 + \zeta_1 + \zeta_2)}.
\]

The last two terms amount the effects caused by the limiting zero \(\zeta^*\). Since

\[
\lim_{T \to 0} g_1 = \lim_{T \to 0} T\zeta^* + \zeta = \lim_{\zeta \to \infty} z - \zeta^* = -1,
\]

\[
\lim_{T \to 0} g_2 = 1
\]

hold, we obtain

\[
\lim_{T \to 0} E_{\delta n}^1 = 2z \left[ 1 - A_{\text{co}}(z) \prod_{k=1}^{n_p} \frac{z + \bar{p}_k}{z - p_k} \right]^2,
\]

which shows that it converges to the corresponding continuous-time term. Furthermore, we see that \(\lim_{T \to 0} E_{\delta n}^2 = 0\) from Case 1. Lastly, we have

\[
\lim_{T \to 0} E_{\delta n}^3 = 0,
\]

since

\[
\lim_{T \to 0} \frac{-2(T\zeta^* + 2\zeta)(T\zeta^* + 2\zeta^*)}{(T\zeta^* + \zeta + \zeta^*)} = \lim_{\zeta \to \infty} \frac{4z(\eta^* + 1)\zeta^*}{\zeta^* + z\eta^*} = 4z(\eta^* + 1),
\]

\[
\lim_{T \to 0} \gamma_1 = 1 - A_{\text{co}}(z) \prod_{k=1}^{n_p} \frac{z + \bar{p}_k}{z - p_k},
\]

\[
\lim_{T \to 0} \gamma_2 = \lim_{\zeta \to \infty} \left[ 1 - A_{\text{co}}(\zeta^*) \prod_{k=1}^{n_p} \frac{p_k(\eta^* + 1) + \zeta^* + \bar{p}_k}{\zeta^* - p_k} \right] = 0.
\]
Therefore, \( \lim_{T \to 0} E_{\delta_n}^2 = 0 \) and \( \lim_{T \to 0} E_{\delta_n}^3 = 0 \) reveal that the limiting zero \( \zeta^* \) does not give any effect when the sampling time tends to zero.

**Case 3:** The only non-minimum phase zeros of \( P_T(\delta) \) are limiting zeros \( \zeta^*_i \in \mathbb{N}_\zeta^* \). The continuous-time counterpart of this case is that the plant \( P_c(s) \) is minimum phase with relative degree more than 4. Immediately we have

\[
\lim_{T \to 0} g_i = \prod_{j \in \mathbb{N}^*_{\zeta^*}, j \neq i} \frac{\eta^*_{i} - \eta^*_{j}}{\eta^*_{j} \eta^*_{i} - 1} =: c^*_i,
\]

\[
\lim_{T \to 0} \gamma_i = \lim_{\zeta^*_i \to \infty} \left[ 1 - A_{co}(\zeta^*_i) \prod_{k=1}^{n_p} \frac{p_k(\eta^*_{i} + 1) + \zeta^*_i + \eta^*_j}{\zeta^*_i - p_k} \right] = 0,
\]

\[
\lim_{T \to 0} \frac{(T|\zeta_i|^2 + 2 \Re\zeta_i)(T|\zeta_j|^2 + 2 \Re\zeta_j)}{g_i g_j (T\zeta_i \zeta_j + \zeta_i + \zeta_j)} = \lim_{\zeta^*_i \to \infty} \left( \frac{\eta^*_i + 1}{\eta^*_i - 1} \right) \frac{\eta^*_j + 1}{\eta^*_j - 1}.
\]

Consequently,

\[
\lim_{T \to 0} E_{\delta_n} = 0
\]

under L’Hopital’s rule.

Cases 1–3 represent the all situation may occur in general, where there exist interaction within usual zeros and limiting zeros, and those among limiting zeros. We have shown that the limiting zeros do not give effects on the optimal performance.

Now we pick one example for illustrating the delta domain result and its convergence.

**Example 4.6** Reconsider the SISO continuous-time plant given in Example 4.3. Implementation of zero-order hold operation yields the corresponding delta domain plant

\[
P(\delta) = \frac{c(\delta - \zeta)}{(\delta - \rho_1)(\delta - \rho_2)},
\]

which has also one non-minimum phase zero at \( \delta = \zeta \) and possibly two unstable poles at \( \delta = \rho_1 \) and \( \delta = \rho_2 \). Under the corresponding weighting functions \( W_s(\delta) \) and \( W_y(\delta) \), Fig. 4.9 shows the computation of \( E^*_\delta \) by using Theorem 4.2 (solid line) and that of \( E^*_c \) by using Theorem 4.7 for sampling time \( T = 0.10, 0.05 \) seconds (dashed and dash-dotted lines, respectively). This confirms our result that \( E^*_\delta \) converges to \( E^*_c \) as \( T \) approaches zero.

### 4.5 Delay-time Case

In this section we provide a tiny contribution on the regulation problem of pure delay-time systems. First we present an implication of our preceding result on energy regulation problem of delta domain case.
Corollary 4.5 Suppose that $P(\delta)$ has relative degree $v$, common non-minimum phase zeros $\zeta_i$ ($i \in \mathbb{N}_\zeta$), and has only one unstable pole $\rho$. Then, the optimal energy regulation performance is given by

$$E_\delta^* = (T \rho + 1)^{2(v-1)} (T \rho^2 + 2 \rho) \left[ \prod_{i \in \mathbb{N}_\zeta} \frac{T \zeta_i \rho + \zeta_i + \rho}{\zeta_i - \rho} \right]^2.$$ 

Proof. This is the delta domain counterpart of Corollary 4.3. Use the relation $\lambda = T \rho + 1$, $\eta = T \zeta + 1$, and note that $E_\delta^* = E_\delta^*/T$.  

From now on, we study the energy regulation problem of continuous-time delay systems. We consider the regulation setup depicted by Fig. 4.10, where the SISO plant $P(s)$ has a pure delay-time in the input port:

$$P(s) = \frac{P_0(s)}{s-p} e^{-\tau s}$$

(4.42)

with $\tau \geq 0$ is the delay time. Note that we consider a simple plant $P(s)$ which has only one unstable pole $p \in \mathbb{C}_+$. $P_0(s)$ is stable and has non-minimum phase zeros $z_i$ ($i = 1, \ldots, n_z$).

We formulate and minimize the following performance index

$$\text{Fig. 4.10. The regulation setup for delay systems.}$$
4.5 Delay-time Case

\[ E_c := \int_0^\infty |u(t)|^2 \, dt \]  

(4.43)

with respect to an impulse disturbance input \( d(t) \). We provide the analytical closed-form expression of the minimal energy regulation performance \( E_c^* \) in the following proposition.

**Proposition 4.1** Let the plant \( P(s) \) is given in (4.42). Then, the optimal energy regulation performance is given by

\[ E_c^* = 2p e^{2p\tau} \left[ \prod_{i=1}^{n_z} \frac{z_i + p}{z_i - p} \right]^2. \]

(4.44)

*Proof.* We follow an indirect way to prove Proposition 4.1, i.e., by using continuity property of delta domain expression. It is known that the zero-order hold operation with sampling time \( T \) will convert the continuous-time delay plant \( P(s) \) given in (4.42) onto its delta domain counterpart \( P(\delta) \) as follows:

\[ P(\delta) = \frac{P_0(\delta)}{\delta - \rho} (T\delta + 1)^{-\tau/T}, \]

where \( P_0(\delta) \) is stable and has non-minimum phase zeros \( \zeta_i (i = 1, \ldots, n_z) \), and \( \rho \) is the unstable pole of \( P(\delta) \). Note that \( \tau/T \) relative degrees are contributed by the discretization of the delay part, while 1 relative degree is from that of \( P_0(s) \). The optimal performance is then can be obtained by application of Corollary 4.5, that is

\[ E_\delta^* = (T\rho + 1)^{2\tau/T} (T\rho^2 + 2\rho) \left[ \prod_{i=1}^{n_z} \frac{T\zeta_i \rho + \zeta_i + \rho}{\zeta_i - \rho} \right]^2. \]

By facts that \( \rho = (e^{pT} - 1)/T \) and \( \zeta_i = (e^{z_i T} - 1)/T \), then we immediately have

\[ \lim_{T \to 0} E_\delta^* = 2p e^{2p\tau} \left[ \prod_{i=1}^{n_z} \frac{z_i + p}{z_i - p} \right]^2. \]

Hence, by the continuity property we can derive the energy regulation performance for delay-time system (4.42).

*Remark:* Proposition 4.1 tells that the time-delay gives its effects in exponential way as well as the unstable pole. It also admits that unstable pole and non-minimum phase zero which close each other generally worsen the regulation performance. Furthermore, if \( z_i = \infty \) then \( E^* = 2p e^{2p\tau} \), which confirms the result in [5]. Additionally if \( \tau = 0 \), i.e., we consider an LTI case, then

\[ E^* = 2p \left[ \prod_{i=1}^{n_z} \frac{z_i + p}{z_i - p} \right]^2, \]
which can be confirmed by Corollary 4.1.

It is well-known that Padé approximation can be used to approximate the delay-time part. The first-order approximation provides

\[ e^{-\tau s} \approx \frac{2}{2/\tau + s}. \]  

(4.45)

Hence, for minimum phase case of \( P_0(s) \), we obtain

\[ P(s) \approx P_{\text{Padé}}(s) = \frac{P_0(s) 2/\tau - s}{s - p 2/\tau + s}. \]

Note that \( P_{\text{Padé}}(s) \) has not only one unstable pole \( p \) but also one extra non-minimum phase zero \( 2/\tau \). It can be calculated by Theorem 4.1 that the optimal energy regulation performance of \( P_{\text{Padé}}(s) \) is given by

\[ E^*_c = 2p \left[ \frac{2/\tau + p}{2/\tau - p} \right]^2. \]

(4.46)

Furthermore, Taylor expansion gives

\[ E^*_c = 2p e^{p\tau} \approx 2p + 4\tau p^2 + 4\tau^2 p^3 + \cdots, \]

and

\[ E^*_{\text{Padé}} = 2p \left[ \frac{2/\tau + p}{2/\tau - p} \right]^2 \approx 2p + 4\tau p^2 + 4\tau^2 p^3 + \cdots, \]

which show that Padé approximation works well only for the small values of \( p \) and \( \tau \). We confirm this fact by example as follows.

**Example 4.7** We consider the following delay-time plant

\[ P(s) = e^{-0.1s}. \]

With delay time \( \tau = 0.1 \text{sec.} \), the Padé approximation plant \( P_{\text{Padé}}(s) \) has one non-minimum phase zero at \( s = 20 \). Hence,

\[ E^*_c = 2p e^{0.2p}, \]

\[ E^*_{\text{Padé}} = 2p \left[ \frac{20 + p}{20 - p} \right]^2. \]

Figs. 4.11 and 4.12 plot these two optimal performances with respect to unstable pole location \( p \). It is shown that for bigger \( p \), \( E^*_{\text{Padé}} \) will be unbounded whenever \( p \) is getting closer to 20 since it happens almost unstable pole-zero cancellation. Two calculations are very close each other but only for small \( p \).
4.6 Summary

In this chapter we have examined the $H_2$ regulation performance problem in SIMO LTI feedback control system. We have formulated and solved the problem in minimizing the energy of the system output under the control input and sensitivity constraints against the impulsive disturbance input. We provide the analytical closed-form expression of the optimal regulation performance in terms of the dynamics and structure of the plant. Toward the existing result of continuous-time case, we have extended the problem to a non-minimum phase system. While in discrete-time case, we have provided new results.

Our result shows that, in discrete-time case, the contribution of unstable poles and plant gain are given in product and exponential ways. This
Regulation Performance Limitations differs from the continuous-time case, where those are given in summation and plain ways. The difference is caused by a fact that the derivation of the expressions for both two cases are completely different. In this process we have to exploit certain functions evaluated at infinity, which is laid on the $j\omega$-axis (boundary of $s$-domain) but not on the unit circle (boundary of $z$-domain). It means that derivation process in discrete-time case is not parallel with that of continuous-time case. This is contrast with the tracking problem, where the derivations for the discrete-time are almost parallel to those for the continuous-time case.

Additionally, we have reformulated and resolved the regulation problem in delta domain by means the delta operator. Analysis on continuity property shows that we can recover the continuous-time expression from the delta domain expression stand point by taking the sampling time to zero. Based on the delta domain expression, we have also derived the analytical closed-form expression of the optimal energy regulation performance for pure delay-time system.

In general, we show that the optimal regulation performance is properly characterized by the plant’s unstable poles and common non-minimum phase zeros, plant gain, and the outer factor. The last can be fairly explained by Poisson-Jensen formula.
5

Applications

This chapter is devoted to application issues. We demonstrate how to apply the analytical closed-form expressions to some physical systems, namely three-disk torsional system, inverted pendulum system, and magnetic bearing system.

5.1 Three-disk Torsional System

5.1.1 Problem Setting

Fig. 5.1 depicts the three degree of freedom torsional apparatus, where the system has three disks of inertia $J_1$, $J_2$, $J_3$, respectively, due to the disks themselves and the masses we affix to the disks, and damping coefficients $c_1$, $c_2$, $c_3$ due to friction in the bearings and other components supporting the disks. A torsional rod connecting all three disks with torsional spring constants $k_1$ and $k_2$ in between the disks. The actuation device, a high torque brushless DC servo motor, applies torque to the bottom disk. The control input $u$ is the voltage applied to the control motor. As feedback sensors, high resolution optical encoders are attached at each disk to measure the angular displacements $\theta_1$, $\theta_2$, and $\theta_3$.

The equations of motion for the three-disk torsional system are as follow:

![Fig. 5.1. Three-disk torsional system.](image-url)
We denote by $P_1(s)$, $P_2(s)$, and $P_3(s)$, the transfer functions from applied voltage $u$ to regulated outputs $\theta_1$, $\theta_3$, and $\theta_3$, respectively. Then they are given by 6th order models

$$
P_1(s) = \frac{N_1(s)}{sD(s)}, \quad P_2(s) = \frac{N_2(s)}{sD(s)}, \quad P_3(s) = \frac{N_3(s)}{sD(s)}, $$

where

$$
N_1(s) = J_1 J_2 J_3 s^4 + (J_2 c_3 + J_3 c_2) s^3 + (J_2 k_2 + c_2 c_3 + J_3 k_1 + J_3 k_2) s^2 + (c_2 k_2 + c_3 k_1 + c_3 k_2) s + k_1 k_2,
$$

$$
N_2(s) = k_1 (J_3 s^2 + c_3 s + k_2),
$$

$$
N_3(s) = k_1 k_2,
$$

and

$$
D(s) = J_1 J_2 J_3 s^5 + (J_1 J_2 c_3 + J_1 J_3 c_2 + J_2 J_3 c_1) s^4 +
\left[ J_1 (J_2 k_2 + J_3 k_1 + J_3 k_2 + c_2 c_3) + J_2 (J_3 k_1 + c_1 c_3) + J_3 c_2 c_1 \right] s^3 +
\left[ J_1 (c_2 k_2 + c_3 k_1 + c_3 k_2) + J_2 (c_1 k_2 + c_3 k_1) + J_3 (c_1 k_1 + c_1 k_2 + c_2 k_1) + c_1 c_2 c_3 \right] s^2 +
\left[ (J_1 + J_2 + J_3) k_1 k_2 + c_1 (c_2 k_2 + c_3 k_1 + c_3 k_2) + c_2 c_3 k_1 \right] s +
(c_1 + c_1 + c_1) k_1 k_2.
$$

Expression for $D(s)$ requires the measurement or estimation of eight dynamic parameters for its coefficients. The more practical forms expressed in terms of measurable frequencies, damping, and gains, are given by

$$
P_1(s) = \frac{K_1 (s^2 + 2 \zeta_1 \omega_n s + \omega_n^2)(s^2 + 2 \zeta_2 \omega_n s + \omega_n^2)}{s(s + \omega_p)(s^2 + 2 \zeta_p \omega_p s + \omega_p^2)},
$$

$$
P_2(s) = \frac{K_2 (s^2 + 2 \zeta_2 \omega_n s + \omega_n^2)}{s(s + \omega_p)(s^2 + 2 \zeta_p \omega_p s + \omega_p^2)},
$$

$$
P_3(s) = \frac{K_3}{s(s + \omega_p)(s^2 + 2 \zeta_p \omega_p s + \omega_p^2)},
$$

Suppose that we fix the parameters $J_1 = 0.002713$, $J_2 = 0.001802$, and $J_3 = J_2$ (all in kg m$^2$), $c_1 = 0.008778$, $c_2 = 0.001834$, and $c_3 = c_2$ (all in kg m/s$^2$), and $k_1 = 3.0568$ and $k_2 = 2.6570$ (all in kg m/s). Then the transfer functions are all (marginally) stable and minimum phase. The location of the poles and zeros is depicted by Fig. 5.2. Note that in this case all the plants have the same set of poles.
5.1 Three-disk Torsional System

5.1.2 Tracking Performance Limitation

In this part we apply our analytical expressions of the optimal tracking performance to the torsional system. We consider the tracking error problem under control input penalty (See Subsection 3.2.3), i.e., we compute

\[ J^*_c := \inf_{K \in \mathcal{K}} \int_0^\infty \left( |e(t)|^2 + |u_w(t)|^2 \right) \, dt, \]

where \( u_w(t) = \mathcal{L}^{-1}\{W_u(s) \hat{u}(s)\} \). Note that the plants all have an integrator, thus the required assumptions are satisfied.

Since the plants are (marginally) stable and minimum phase, then the optimal tracking performances are given by Corollary 3.2 as follow:

\[ J^*_{c,i} = \frac{1}{\pi} \int_0^\infty \log \left[ 1 + \frac{|W_u(j\omega)|^2}{|P_i(j\omega)|^2} \right] \frac{d\omega}{\omega^2}, \quad (5.1) \]

for \( i = 1, 2, 3 \). Fig. 5.3 plots the optimal performance \( J^*_{c,i} \) for \( W_u \) from 0 to 1. It is shown that disk 1 provides the best tracking performance and disk 3 gives the worst.

5.1.3 Regulation Performance Limitation

In this subsection we apply our analytical expressions of the optimal regulation performance to the torsional system. We consider the output regulation problem, where we minimize the energy of control input jointly with that of system output (See Subsection 4.2.2), i.e., we compute

\[ E^*_c := \inf_{K \in \mathcal{K}} \int_0^\infty \left( |y_w(t)|^2 + |u(t)|^2 \right) \, dt, \]
where \( y_w(t) = L^{-1}\{W_y(s)\hat{y}(s)\} \). Note that the plants are all strictly proper, thus the required assumption is satisfied.

Since the plants are (marginally) stable and minimum phase, then the optimal regulation performances are given by Theorem 4.2 as follow:

\[
E^*_c, i = \frac{1}{\pi} \int_0^{\infty} \log (1 + |W_y(j\omega) P_i(j\omega)|^2) \, d\omega, \quad (5.2)
\]

for \( i = 1, 2, 3 \). Fig. 5.4 plots the optimal performance \( E^*_c, i \) for different ranges of \( W_y \). Differ to tracking performance case, regulation performance problem
5.2 Inverted Pendulum System

5.2.1 Problem Setting

For an illustration of our results on tracking performance limitations we consider the inverted pendulum system shown in Fig. 5.5, where an inverted pendulum is mounted on a motor driven-cart. We assume that the pendulum moves only in the vertical plane, i.e., two dimensional control problem. We here assume that $M$, $m$, and $2l$ respectively denote the mass of the cart, the mass of the pendulum, and the length of the pendulum. We also assume that the friction between the track and the cart is $\mu_t$ and that between the pendulum and the cart is $\mu_p$. We consider an uniform pendulum so that its inertia is given by $I = \frac{1}{3}ml^2$.

The equations of motion between the control input $u$, which is the force to the cart, the position of the cart $x$, the angle of the pendulum $\theta$ are represented by

$$
\begin{align*}
\frac{4}{3}ml^2\ddot{\theta} + \mu_p \dot{\theta} - mgl\theta &= -ml\ddot{x}, \\
(M + m)\ddot{x} + \mu_t \dot{x} - ml\ddot{\theta} &= u,
\end{align*}
$$

under the assumption that the angle $\theta$ is small. Taking the Laplace transform of the system equations yields

$$
\begin{align*}
\frac{4}{3}ml^2\Theta(s)s^2 + \mu_p \Theta(s)s - mgl\Theta(s) &= -mlX(s)s^2, \\
(M + m)X(s)s^2 + \mu_t X(s)s - ml\dot{\Theta}(s)s^2 &= U(s).
\end{align*}
$$

Then, the transfer functions from $u$ to $x$ (denoted by $P_x$), from $u$ to $\theta$ (denoted by $P_\theta$), and that from $u$ to the position of the edge of the pendulum $x + 2l\theta$ (denoted by $P_{x\theta}$) are, respectively, given by

![Fig. 5.5. The inverted pendulum system.](image-url)
\[
    P_x(s) = \frac{4ml^2s^2 + \mu_p s - mgl}{sD(s)},
\]
\[
    P_\theta(s) = \frac{-mls}{D(s)},
\]
\[
    P_{x\theta}(s) = \frac{-\frac{2}{3}ml^2s^2 + \mu_p s - mgl}{sD(s)},
\]
where
\[
    D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0,
\]
with \(a_3 := \frac{1}{4}(4M + m)ml^2\), \(a_2 := (M + m)\mu_p + \frac{4}{3}\mu_t ml^2\), \(a_1 := -(M + m)mgl + \mu_p \mu_t\), and \(a_0 := -\mu_t mgl\). Note that all of the plants have a common unstable pole at \(p\). Additionally, \(P_x(s)\) has a non-minimum phase zero at
\[
    z_x = \frac{-3\mu_p + \sqrt{9\mu_p^2 + 48m^2 gl^3}}{8ml^2},
\]
and \(P_{x\theta}(s)\) has two non-minimum phase zeros at
\[
    z_{x\theta} = \frac{3\mu_p \pm \sqrt{9\mu_p^2 - 24m^2 gl^3}}{4ml^2}.
\]

To study the tracking performance of inverted pendulum system, we consider three different cases:

C1: The control objective in this case is to control the cart position \(x(t)\), i.e., we consider an SISO plant \(P(s) = P_x(s)\).

C2: We shall control the cart position \(x(t)\) and the pendulum angle \(\theta(t)\), i.e., we consider an SITO plant
\[
    P(s) = \begin{bmatrix} P_x(s) \\ P_\theta(s) \end{bmatrix}.
\]
We select \(\nu = (1, 0)^T\) as the step input direction.

C3: In this case we control the cart position \(x(t)\) and the position of the edge of the pendulum \(x + 2l\theta\), i.e., we consider an SITO plant
\[
    P(s) = \begin{bmatrix} P_x(s) \\ P_{x\theta}(s) \end{bmatrix}.
\]
We choose \(\nu = (1, 1)^T\) as the step input direction.

5.2.2 Tracking Error Problem

We here give the analytical closed-form expression for the best achievable tracking error performance. We assume \(\mu_p = \mu_t = 0\) for simplicity, i.e., there are no frictions imposed, from which the unstable pole is determined by
\[ p = \sqrt{\frac{3(M + m)g}{(4M + m)l}}, \quad (5.5) \]

We denote by \( J^*_c, J^*_{c2}, \) and \( J^*_{c3} \) the optimal tracking performances for Cases C1, C2, and C3, respectively. According to Theorem 3.2 we obtain

\[
\begin{align*}
J^*_c &= 4 \sqrt{\frac{l}{3g}} \left[ \frac{\sqrt{M + m} + \sqrt{M + \frac{1}{4}m}}{\sqrt{M + m} - \sqrt{M + \frac{1}{4}m}} \right]^2, \\
J^*_{c2} &= 4 \sqrt{\frac{l}{3g}} + \frac{1}{\pi} \int_0^\infty \log \left[ 1 + \frac{9\omega^4}{(4\omega^2 + 3g)^2} \right] \frac{d\omega}{\omega^2}, \\
J^*_{c3} &= 4 \sqrt{\frac{l}{3g}} + \frac{1}{\pi} \int_0^\infty \log \left[ \frac{1}{2} + \frac{1}{2} \frac{1}{(4\omega^2 + 3g)^2} \right] \frac{d\omega}{\omega^2} + \frac{1}{\pi} \int_0^\infty \log \left[ \frac{1}{2} + \frac{1}{2} \frac{1}{(2l\omega^2 - 3g)^2} \right] \frac{d\omega}{\omega^2}.
\end{align*}
\]

We remark that in Case C1, the optimal tracking performance is imposed by both non-minimum phase zero \( z_x \) and unstable pole \( p \). Hence, \( J^*_c \) depends on the mass of the cart \( M \), the mass and the length of the pendulum, \( m \) and \( l \). While in Cases C2 and C3 the unstable pole \( p \) does not give an effect since its direction is not coincident with that of step input signal \( \nu \), i.e., \( \tilde{M}(p)\nu \neq 0 \). Consequently, the optimal performances \( J^*_{c2} \) and \( J^*_{c3} \) depend only on \( l \) but are independent of \( M \) and \( m \).

Furthermore, if we assume that the ratio between the mass and the length of the pendulum is constant, i.e., \( \frac{m}{l} = \psi \) for a real constant \( \psi \), then the length
which gives the lowest possible performance in Case C1 is

\[ l^* = \frac{(\sqrt{265} - 5)M}{2\psi} \]  \hspace{1cm} (5.6)

It can be readily seen that we can make \( l^* \) small by reducing the mass of the cart \( M \) or by increasing the pendulum parameter \( \psi \). We may also rewrite (5.6) as

\[ \frac{m^*}{M} = \frac{\sqrt{265} - 5}{2} \approx 5.6394 \]  \hspace{1cm} (5.7)

which suggests that the lowest possible value of the optimal tracking performance can be achieved as long as the ratio between the mass of the pendulum and that of the cart satisfies (5.7), regardless the type of material we use for the pendulum.

Fig. 5.6 illustrates the relationship between the length of the pendulum and the optimal performance for \( \psi = 2\frac{\pi}{145} \text{kg/m} \) and \( M = 2\text{kg} \). In Case C1, \( l \approx 1.7 \text{m} \) gives the lowest possible performance. While in Cases C2 and C3, shorter pendulum is preferred since it provides smaller tracking performance.

5.2.3 Tracking Error Problem under Control Penalty

This part considers the tracking error problem under control input penalty of inverted pendulum system. We here assume \( \mu_p \neq 0, \mu_t \neq 0, \) and \( W_u(s) = 1 \).

We define the following functions:

\[ f_1(\omega) := \omega^2[(a_0 - a_2\omega^2)^2 + (a_1\omega - a_3\omega^3)^2], \]
\[ f_2(\omega) := m^2\omega^2(\frac{4}{3}l\omega^2 + g)^2 + \mu_p^2\omega^2, \]
\[ f_3(\omega) := m^2\omega^2(\frac{2}{3}l\omega^2 - g)^2 + \mu_p^2\omega^2, \]

and

\[ J_z := \frac{16ml^2}{-3\mu_p + \sqrt{9\mu_p^2 + 48m^2gl^3}}, \]

which denotes the effect caused by the non-minimum phase zero of \( P_x(s) \), i.e., \( z_x \). According to Theorem 3.3, the optimal performances for Cases C1, C2, and C3, are respectively given by

\[ J_{c1}^* = J_z + \frac{1}{\pi} \int_0^\infty \log \left[ 1 + \frac{f_1(\omega)}{f_3(\omega)} \right] \frac{d\omega}{\omega^2} + \frac{2[1 - \Theta_1^{-}(p)\Theta_1(0)]^2}{p}, \]
\[ J_{c2}^* = J_z + \frac{1}{\pi} \int_0^\infty \log \left[ 1 + \frac{f_1(\omega) + m^2\omega^2}{f_2(\omega)} \right] \frac{d\omega}{\omega^2}, \]
\[ J_{c3}^* = J_z + \frac{2\mu_p}{mgl} + \frac{1}{\pi} \int_0^\infty \log \left[ \frac{[f_1(\omega) + f_2(\omega) + f_3(\omega)]^2}{f_2(\omega)f_3(\omega)} \right] \frac{d\omega}{\omega^2}. \]
where $\Theta_1(s)$ in $J_{c1}^*$ is determined from the inner-outer factorization

$$
\begin{bmatrix}
M_x(s) \\
N_x(s)
\end{bmatrix} = \Theta_1(s)\Theta_o(s)
$$

with $P_x(s) = N_x(s)M_x^{-1}(s)$. The existence of $\Theta_1(s)$ makes the analytical formula incomplete since we don’t know the closed-form expression of this transfer function.

Fig. 5.7 depicts the relationship between the length of the pendulum and the optimal performance for $\psi = 2.145\pi\text{kg/m}$, $M = 2\text{kg}$, $\mu_t = \frac{1}{4}$, and $\mu_p = \frac{1}{3}$. We can see from the figure that the values of $J_{c1}^*$ are quite large which are caused by the third term of $J_{c1}^*$, and that the optimal length of the pendulum which gives the lowest possible performance is around 2.8m. The values of optimal costs $J_{c2}^*$ for Case C2 is quite small in comparison with $J_{c1}^*$. The main reason is that the unstable pole $p$ does not give any effect since $p$ is a common pole of $P_x(s)$ and $P_\theta(s)$, and its direction is not coincident with that of step input signal, i.e., $\tilde{M}(p)\nu \neq 0$. In this case, the best length of the pendulum is about 0.2m. Note that in Case C3, the second term is the effect caused by non-minimum zeros of $P_x\theta(s)$, i.e., $z_{x\theta}$, but $J_{c3}^* = \infty$ since the integral term is infinity. The integral term accounts the total variation of the plant direction with frequency. It is understood that a rapid change of plant direction at low frequency will impose a more deteriorate constraint upon the achievable tracking performance.
5.3 Magnetic Bearing System

5.3.1 Problem Setting

In this section we implement our results on regulation performance limitation provided in Chapter 4 to study the performance limitations in a magnetic bearing system, which has been widely investigated, see e.g., [39,56,57].

We consider a simple active magnetic bearing (AMB) depicted in Fig. 5.8. AMBs suspend the levitated object (generally, a rotor) of mass $M$ by forces of two opposing magnetic attractions which are supplied by power switching amplifiers of voltages $V_1, V_2$ and currents $I_1, I_2$. AMBs use actively controlled electromagnetic forces to control the position of the rotor or other ferromagnetic body in air which has nominal air gap $g_0$.

If we assume that the state variable can be forced to track some constant trajectory $\Phi_0$ by appropriate choice of control input $u$, then a linearizing model may be realized as follows [39]:

$$
\frac{d}{dt} \begin{bmatrix} x \\ v \\ \phi \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \phi_0 \\ \mu \phi_0 & 0 & -\mu \end{bmatrix} \begin{bmatrix} x \\ v \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,
$$

$$
y_c = \begin{bmatrix} -\phi_0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x \\ v \\ \phi \end{bmatrix}, \quad y_p = x,
$$

where $x$ and $v$ respectively denote the normalized position of the rotor and its derivative, and $\phi$, $u$, and $y_c$ are respectively normalized differences of the fluxes ($\Phi_1, \Phi_2$), input voltages ($V_1, V_2$), and output currents ($I_1, I_2$) of left and right magnetics, which are given by

$$
\phi := \frac{\Phi_1 - \Phi_2}{A_g B_{sat}}, \quad u := \frac{V_1 - V_2}{V_0}, \quad y_c := \frac{I_1 - I_2}{I_{sat}}
$$

with appropriate constant $A_g$, $B_{sat}$, $V_0$, and $I_{sat}$.

We define by $P_c$ the transfer function from the control input to the current sensor, i.e., from $u$ to $y_c$, by $P_p$ the transfer function from the control input to the position sensor, i.e., from $u$ to $y_p$, and by $P_{cp}$ the transfer function from
the control input to both current and position sensors, i.e., from $u$ to $y_c$ and $y_p$. In other words, we define

$$P_c(s) = \frac{s^2 - \phi_0^2}{s^3 + \mu s^2 - \mu \phi_0^2},$$

$$P_p(s) = \frac{\phi_0}{s^3 + \mu s^2 - \mu \phi_0^2},$$

$$P_{cp}(s) = \begin{bmatrix} P_c(s) \\ P_p(s) \end{bmatrix}.$$  

It is clear that $P_c(s)$ has one non-minimum phase zero at $\phi_0$ and one unstable pole $p$ lying between 0 and $\phi_0$, depending on the value of $\phi_0$ and $\mu$. It is known that physically reasonable value of $\sigma := \phi_0/\mu$ ranges between about 0.3 and 3 and that, the unstable pole $p$ ranges from about $0.6\phi_0$ and $0.9\phi_0$ [39]. Further, $P_p(s)$ has no non-minimum phase zero but one unstable pole at $p$. Meanwhile, $P_{cp}(s)$ has no common non-minimum phase zero but one unstable pole, also at $p$.

We here examine the regulation performance limitation of the AMB as measured by the $\mathcal{H}_2$ norm. In other words, we minimize the following performance measure

$$E_c = \int_0^\infty \left( |y_{w1}(t)|^2 + \rho_c |y_c(t)|^2 + \rho_p |y_p(t)|^2 + |u(t)|^2 \right) \, dt, \quad (5.8)$$

where $\rho_c$ and $\rho_p$ are some constants. Indeed, the above performance index reveals the most natural setting in the control application, where $y_{w1}$ accounts the sensitivity measure, $y_c$ and $y_p$ quantify the regulation outputs, and control input $u$ represents the disturbance attenuation. In practical situation, the current should be bounded. Thus, include the current sensor $y_c$ into the performance index to be minimized is a more realistic setup.

We examine the following three cases relate to the choice of the weighting function $W_y$:

C1: For $P(s) = P_c(s)$,

$$W_{y,c}(s) = \begin{bmatrix} \rho_c \\ \rho_p \end{bmatrix}.$$  

C2: For $P(s) = P_p(s)$,

$$W_{y,p}(s) = \begin{bmatrix} \frac{\rho_c (s+\phi_0)^2}{\phi_0^3} \\ \frac{\rho_p}{\phi_0} \end{bmatrix}.$$  

C3: For $P(s) = P_{cp}(s)$,

$$W_{y,cp}(s) = \begin{bmatrix} \rho_c & 0 \\ 0 & \rho_p \end{bmatrix}.$$
Note that the definitions $W_y(s)$ above assure that the performance indexes for the three cases are completely the same, i.e.,

$$W_{y,c}(s)P_c(s) = W_{y,p}(s)P_p(s) = W_{y,cp}(s)P_{cp}(s).$$

By letting the weighting function $W_y$ depend on parameters $\rho_p$ and $\rho_c$ we gain more degree of freedom. We immediately can assign $\rho_c = 0$ such that we regulate only the position $y_p$. For all the cases we choose the weighting function $W_s$ as

$$W_s(s) = \frac{w}{1 + \tau s},$$

where $\tau$ is the bandwidth and $w$ is a real constant.

To facilitate our analysis, we define by $E^\ast_{c,c}$, $E^\ast_{c,p}$, $E^\ast_{c,cp}$, the optimal performances correspond to $P_c(s)$, $P_p(s)$, and $P_{cp}(s)$, respectively.

### 5.3.2 Regulation Problem: Continuous-time Case

We first note that the first terms of (4.17), $E_{cn}$, are the same for all the three cases. Since $P_p(s)$ has no non-minimum phase zero and $P_{cp}(s)$ has no common one (note that only common non-minimum phase zero gives limitation), we can see that $E_{cn} = 0$ and hence the optimal performance of these two cases are equal. On the other hand, $P_c(s)$ has one non-minimum phase zero at $\Phi_0$, and hence $E_{cn}$ is always positive. This observation implies that the following relations generally hold:

$$E^\ast_{c,c} > E^\ast_{c,p} = E^\ast_{c,cp}, \quad (5.9)$$

where

$$E^\ast_{c,s} - E^\ast_{c,p} = 2\Phi_0 \left[1 - A_o(\Phi_0) \frac{\Phi_0 + p}{\Phi_0 - p}\right]^2.$$ 

This can be confirmed by the following further investigation. First we consider a case where $\rho_c = \rho_p = 0$, i.e., $W_y(s) = 0$. For this special case, clearly we obtain

$$A_o(s) = \frac{\sqrt{1 + w^2 + \tau \Phi_0}}{1 + \tau \Phi_0}.$$ 

Then, the closed-form expression of the optimal performances can be expressed as

$$E^\ast_{c,p} = E^\ast_{c,cp} = 2p + \frac{1}{\pi} \int_0^\infty \log \left[1 + \frac{w^2}{1 + \omega^2 \tau^2}\right] d\omega = 2p + \frac{\sqrt{1 + w^2} - 1}{|\tau|}$$

and

$$E^\ast_{c,c} - E^\ast_{c,cp} = 2\Phi_0 \left[1 - \frac{\sqrt{1 + w^2 + \tau \Phi_0} \Phi_0 + p}{1 + \tau \Phi_0} \frac{\Phi_0 - p}{\Phi_0}\right]^2 > 0.$$
5.3 Magnetic Bearing System

We now compute the optimal performances with the following physical parameters [39]: $\Phi_0 = 0.288$, $\mu = 0.582$, from which we get $p = 0.242$. For the weighting function $W(s)$, we take $w = 1$ and $\tau = 1$. The computation results give $E^*_{c,c} = 117.7013$ and $E^*_{c,p} = E^*_{c,cp} = 0.8983$. The complete calculations for different value of $\mu$ are depicted by Figs. 5.9(a) and 5.9(b). We also provide the numerical calculation by using Matlab toolbox. The calculations confirm the equality and inequality relations (5.9).

Next, we consider a case where $\rho_c = \rho_p = 1$. Note that in computation $E^*_{c,cr} A_o(s)$ is determined from the inner-outer factorization

$$\begin{bmatrix}
W_c(s) \\
W_{y,c}(s)N_c(s) \\
-1
\end{bmatrix} = A_i(s)A_o(s),$$

where $N_c(s)$ is the coprime factor of $P_c(s)$, i.e.,

$$P_c(s) = N_c(s)M_c^{-1}(s).$$

The computation results provide $E^*_{c,c} = 716.5626$ and $E^*_{c,p} = E^*_{c,cp} = 1.5821$. One physical interpretation can be gained from relations (5.9) is that putting extra sensor does not always conduce an performance improvement.

5.3.3 Regulation Problem: Discrete-time Case

We here discuss the discrete-time case. We assume that the corresponding discrete-time transfer functions $P_c(z)$, $P_p(z)$, $P_{cp}(z)$ are obtained from the zero-order hold operations of $P_c(s)$, $P_p(s)$, $P_{cp}(s)$, respectively. By these operations, we know that $P_c(z)$ has two non-minimum phase zeros at 1.3351 and at infinity. $P_p(z)$ has also two non-minimum phase zeros at $-3.2498$ and at infinity, while $P_{cp}(z)$ has no common non-minimum phase zero except at
infinity. All the plants have one unstable pole at 1.2738. The optimal performance then can be computed based on Theorem 4.4, and we can show the following relations:

\[ E_{d,c}^* > E_{d,p}^* > E_{d,cp}^*. \] (5.10)

For example, for \( \rho_c = \rho_p = 0 \) with the sampling time \( T = 1 \) second, we have \( E_{d,c}^* = 150.3615 \), \( E_{d,p}^* = 1.7506 \), and \( E_{d,cp}^* = 1.3383 \), which implies that using multiple sensors has an advantage for the discrete-time system. For the complete calculations of different value of \( \mu \), see Figs. 5.10(a), 5.10(b), and 5.10(c). The calculations confirm the inequality relations (5.10). However, the differences between \( E_{d,p}^* \) and \( E_{d,cp}^* \) become smaller when the sampling time \( T \) is reduced, as shown in Fig. 5.10(d).
We have examined the $\mathcal{H}_2$ optimal tracking and regulation problems in single-input multiple-output (SIMO) linear time-invariant (LTI) feedback control systems. Instead of seeking the optimal controllers, we have been deriving the analytical closed-form expressions of the optimal performances in terms of the plant dynamics and structures. In other words, we have quantified and characterized the fundamental performance limitations arise in the $\mathcal{H}_2$ optimal control problem and provided guidelines for plant design from the view point of control. We have provided the comprehensive and unified solutions to the problem since we derive the analytical expressions for three domains: continuous-time, discrete-time, delta-time systems, and show their unification.

We rely our derivation on the factorization approach such as the coprime factorizations of the plant which led to the Bezout identity and the Youla parameterization of the all stabilizing controllers, and the inner-outer factorization of transfer function. By deeply exploiting this approach,

(i) We have provided the analytical closed-form expressions of the optimal performance in tracking step input signal. The tracking ability itself is measured by the error between the input reference signal and the measurement output, possibly under control input constraint. In $\mathcal{H}_2$ optimal control setting, the tracking performance $J$ is given by

\[
J = \|\hat{e}(\cdot)\|^2_2, \\
J = \|\hat{e}(\cdot)\|^2_2 + \|W_u(\cdot)\hat{u}(\cdot)\|^2_2,
\]

where $W_u$ is the stable and minimum phase weighting function. Briefly, we found that

- **Continuous-time Case:** The available continuous-time results [12] and [31] contain small mistake. We correct the mistake by explicitly accounting an additional effect $J_{\text{c1}}$ caused by the plant’s unstable poles $p_k$. However, the additional term $J_{\text{c1}}$ is equal to zero when-
ever the plant $P$ is either scalar or SIMO with the set of all unstable poles of $P_i$ ($i = 1, \ldots, m$) are completely the same.

- **Discrete-time Case:** The derivation process of the discrete-time case is almost similar to that of the continuous-time case. The key idea proposed in this way is the application of the bilinear transformation, from which we enable to derive two key lemmas from the continuous-time counterparts.

- **Delta Domain Case:** The derivation of the delta-time expressions including two key lemmas can be obtained in straightforward manner by variables substitution. Differ to the discrete-time case, the expressions in the delta-time case contain the sampling time $T$. We demonstrated that the continuous-time expressions can be completely recovered by approaching $T$ to zero. In this regard, we show that the limiting zeros of the discretized plant do not give any effects to the performance provided the sampling time is sufficiently small.

- **Delay-time Case:** By exploiting the delta domain expressions we confirm that the time delay degrades the tracking performance in much the same manner as non-minimum phase zeros.

- **Sampled-data Case:** We have employed an approximation approach by implementing fast sampling technique to derive the similar result of sampled-data feedback control systems. Fast sampling process of sampling time $T/N$ enables us to approximate the sampled data feedback control system to one of discrete-time system. Immediately we can derive the approximation of the optimal performance by invoking the discrete-time result. We have shown by example that we can approximate the exact result quite well. Particularly if the sampling time $T$ is small, $N$ can also be made small.

In general, the optimal tracking performance of SIMO system is explicitly characterized by the plant’s non-minimum phase zeros $\eta_{ij}$ and unstable poles $\lambda_k$, the plant direction which mostly determined by the plant gain, and the reference input direction $\nu$. Furthermore, problem of minimizing the tracking error under control input penalty provides additional limits imposed by $W_u$, which appears in the logarithmic term and inner factor. If we set $W_u = 0$ then we can easily obtain the non-penalty result.

(ii) We have provided the analytical closed-form expressions of the optimal regulation performance against an impulsive disturbance input $d$. In this problem the regulation performance is measured by the energy of the control input $u$, possibly under system output $y_w := (y_{w1}^T, y_{w2}^T)^T$ and sensitivity constraints. In $H_2$ optimal control setting, the regulation performance $E$ is given by

$$E = ||\hat{u}(\cdot)||_2^2,$$

$$E = ||\hat{y}_{w1}(\cdot)||_2^2 + ||\hat{y}_{w2}(\cdot)||_2^2 + ||\hat{u}(\cdot)||_2^2,$$

where $\hat{y}_{w1}$ measures the sensitivity reduction and $\hat{y}_{w2}$ accounts the regulated output. We briefly summarize as follows.
- **Continuous-time Case:** We have completed the results in continuous-time system by deriving the analytical closed-form expression for non-minimum phase system and have extended it by considering a problem with sensitivity penalty.

- **Discrete-time Case:** We have provided the first result on the regulation performance limitation of the discrete-time systems. Differ to the tracking problem, the derivation process of the optimal regulation performance for discrete-time case is not parallel with that of continuous-time case since we have to exploit a certain function evaluated at infinity which is laid on the $j\omega$-axis (boundary of $s$-domain) but not on the unit circle (boundary of $z$-domain). Consequently, the unstable poles of discrete-time system give their effects in exponential way, while those of continuous-time systems contribute their effect in linear manner.

- **Delta Domain Case:** We have demonstrated that the continuous-time expressions can be completely recovered from the delta-time expressions stand-point by approaching the sampling time $T$ to zero. We also show that the limiting zeros do not contribute any effects on the optimal regulation performance.

- **Delay-time Case:** We have provided the analytical closed-form expression of the optimal energy regulation performance for simple SIMO delay-time system, where the plant is non-minimum phase and has only single unstable pole. The derivation has been carried-out by invoking discrete-time and delta-time expressions of the corresponding problem. This tiny contribution shows that the time delay as well as the unstable pole give effects in exponential manner.

Generally speaking, the optimal regulation performance of SIMO system is explicitly characterized by the plant’s non-minimum phase zeros $\eta_{ij}$ and unstable poles $\lambda_k$ as well as the plant gain. It is important to point-out that only the common non-minimum phase zeros influence the performance. Furthermore, if we set $W_s = W_y = 0$ then we obtain the optimal energy regulation performance, i.e., the problem without penalty.

(iii) We have confirmed the effectiveness of the derived expressions by several illustrative examples. We have also shown how to apply the analytical expressions to practical applications including the control of three-disk torsional system, the determination of the optimal parameters in inverted pendulum system, and the selection of the sensor strategy in magnetic bearing system. It has been demonstrated that the analytical closed-form expressions of the optimal performance are quite useful in determining the easily controllable plant.
A

Some Proofs

A.1 Proof of Theorem 3.5

Given the inner-outer factorization (3.44), we define a norm preserving function
\[
\Psi(z) := \begin{bmatrix} \Theta_i(z) \\ I - \Theta_i(z)\Theta_i(z) \end{bmatrix},
\]
i.e., \( \Psi(e^{j\theta})\Psi(e^{j\theta}) = I \). By pre-multiplying \( \Psi(z) \) to (3.45), we obtain
\[
J_d^* = \inf_{Q \in \mathcal{H}_\infty} \left\| \frac{\Theta_i - \Theta_i(1)}{z - 1} \right\|_2^2 + \left\| \frac{(I - \Theta_i\Theta_i)\nu}{z - 1} \right\|_2^2
\]
\[
= \left\| \frac{(\Theta_i - \Theta_i(1))\nu}{z - 1} \right\|_2^2 + \inf_{Q \in \mathcal{H}_\infty} \left\| \frac{\Theta_i(1) + \Theta_o(\tilde{Y} - Q\tilde{M})\nu}{z - 1} \right\|_2^2
\]
\[
= J_1^* + J_2^*,
\]
where
\[
J_1^* := \left\| \frac{(\Theta_i - \Theta_i(1))\nu}{z - 1} \right\|_2^2 + \left\| \frac{(I - \Theta_i\Theta_i)\nu}{z - 1} \right\|_2^2,
\]
\[
J_2^* := \inf_{Q \in \mathcal{H}_\infty} \left\| \frac{\Theta_i(1) + \Theta_o(\tilde{Y} - Q\tilde{M})\nu}{z - 1} \right\|_2^2.
\]
Note that \( J_1^* \) is the optimal performance corresponding to the stable part of the plant, that is \( N(z) \). Since \( N = \Theta_i\Theta_o \), where \( \Theta_i = (w_1, \ldots, w_m)^T \), then \( w_i(z) (i = 1, \ldots, m) \) has the same set of non-minimum phase zeros as \( N_i(z) \).
It is immediate from Lemma 3.4 that
\[
\frac{w_1'(1)}{w_1(1)} = - \sum_{j=1}^{n_1} |\eta_j|^2 - 1 - \sum_{k \in J_n} |\lambda_k|^2 - 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{w_1(\epsilon)}{w_1(1)} \right| \frac{d\theta}{1 - \cos \theta}.
\]

The second term of above equation appears since the set of non-minimum phase zeros of \(N_i(z)\) contains the unstable poles of \(P_i(z), \, i \neq j\). And by noting that \(|w_1(\epsilon)| = |N_i(\epsilon)|/\|N(\epsilon)| = |P_i(\epsilon)|/\|P(\epsilon)|\), we show that

\[
J_1^* = J_{ds1} + J_{ds2} + J_{du1}.
\]

Now we carry on the derivation of \(J_2^*\). We define \(g(z) := \tilde{M}(z)\nu\). Since \(g(\lambda_k) = 0\) for all \(k \in J_p\), and since \(g(z)\) is left invertible, we can factorize it in the form of \(g(z) = m(z)h(z)\), where \(m(z)\) is left invertible in \(\mathbb{R}\mathcal{H}_\infty\) and \(h(z)\) is inner function defined by

\[
h(z) := \prod_{k \in J_p} \frac{z - \lambda_k}{1 - \lambda_k z}.
\]

Consequently,

\[
J_2^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{\Theta^{-}_1(1) + \Theta_o \tilde{Y} - \Theta_o Q_m}{h} \right\|_2^2 = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{\Theta^{-}_1(1) + \Theta_o \tilde{Y} - \Theta_o Q_m}{h} \right\|_2^2.
\]

Let denote \(R_1(z) := \Theta_o(z)\tilde{Y}(z)\). Based on a partial fraction expansion procedure used in \([12, 17, 58]\), it is possible to write

\[
\frac{\Theta^{-}_1(1) + R_1(z)}{h(z)} = \sum_{k \in J_p} \frac{1}{v_k} \frac{\Theta^{-}_1(1) + R_1(\lambda_k)}{h_k} + R_2(z),
\]

where

\[
v_k(z) := \frac{z - \lambda_k}{1 - \lambda_k z},
\]

\[
h_k := \prod_{\ell \in J_p, \ell \neq k} \frac{\lambda_k - \lambda_{\ell}}{1 - \lambda_{\ell} \lambda_k},
\]

and \(R_2(z)\) is in \(\mathbb{R}\mathcal{H}_\infty\). Then,

\[
J_2^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left[ \sum_{k \in J_p} \frac{1}{v_k} \frac{\Theta^{-}_1(1) + R_1(\lambda_k)}{h_k} + R_2 + \Theta_o Q_m \right] \frac{1}{z - 1} \right\|_2^2 = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left[ \sum_{k \in J_p} \frac{1}{v_k} \frac{1 - \lambda_k}{1 - \lambda_k} \frac{\Theta^{-}_1(1) + R_1(\lambda_k)}{h_k} + R_2 + \Theta_o Q_m \right] \frac{1}{z - 1} \right\|_2^2.
\]
where
\[ R_3(z) = R_2(z) + \sum_{k \in J_p} \frac{\Theta_1^-(1) + R_1(\lambda_k)}{h_k}. \]
Since
\[ \left[ \frac{1}{v_k} - \frac{1 - \bar{\lambda}_k}{1 - \lambda_k} \right] \frac{1}{z - 1} \in \mathcal{H}_2^+, \]
and
\[ (R_3(z)\nu - \Theta_\alpha(z)Q(z)m(z)) \frac{1}{z - 1} \in \mathcal{H}_2, \]
then we have
\[ J_2^* = \left\| \left[ \sum_{k \in J_p} \left[ \frac{1}{v_k} - 1 \right] \frac{\Theta_1^-(1) + R_1(\lambda_k)}{h_k} \right] \frac{1}{z - 1} \right\|^2 + \inf_{Q \in \mathcal{H}_\infty} \left\| (R_3\nu - \Theta_\alpha Qm) \frac{1}{z - 1} \right\|^2. \]
Direct calculation yields
\[ \left[ \frac{1}{v_k} - \frac{1 - \bar{\lambda}_k}{1 - \lambda_k} \right] \frac{1}{z - 1} = \frac{|\lambda_k|^2 - 1}{(z - \lambda_k)(1 - \lambda_k)}. \]
Moreover, since \( \Theta_\alpha \) is right-invertible and \( m \) is left-invertible we may properly select a \( Q \) such that
\[ \inf_{Q \in \mathcal{H}_\infty} \left\| (R_3\nu - CQm) \frac{1}{z - 1} \right\|^2 = 0. \]
Hence, by choosing \( \nu = \Theta_i(1) \) we obtain
\[ J_2^* = \left\| \sum_{k \in J_p} \frac{1}{z - \lambda_k} \frac{(|\lambda_k|^2 - 1)(1 - R_i(\lambda_k)\Theta_i(1))}{h_k(1 - \lambda_k)} \right\|^2. \]
Since \( X\tilde{M} = I + \Theta_iR_1 \) which implies \( \Theta_1^-X\tilde{M} = \Theta_1^- + R_1, \) then we obtain
\( R_1(\lambda_k) = -\Theta_1^- (\lambda_k) \) for all \( k \in J_p. \) Therefore,
\[ J_2^* = \sum_{k, \ell \in J_p} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)}{h_k h_\ell(1 - \lambda_k)(1 - \lambda_\ell)} (1 - \Theta_1^- (\lambda_k)\Theta_i(1))(1 - \Theta_1^- (\lambda_\ell)\Theta_i(1)) \times \left( \frac{1}{z - \lambda_k} \cdot \frac{1}{z - \lambda_\ell} \right). \]
Now we calculate the inner product term. Note that for a given stable plant
\[ P(z) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]

its \( \mathcal{H}_2 \) norm is determined by \( \| P(z) \|_2^2 = D^2 + CLC^T \), where \( L \) is the solution of \( L = ALA^T + BB^T \). Since for \( \lambda \in \mathbb{D} \),

\[ \frac{1}{z - \lambda} = \begin{bmatrix} 1/\lambda & 1 \\ 1/\lambda & 0 \end{bmatrix} \]

is stable, then we obtain

\[ \frac{1}{z - \lambda} = \begin{bmatrix} 1/|\lambda| & 1 \\ 1/|\lambda| & 0 \end{bmatrix} \]

Consequently,

\[ \langle \frac{1}{z - \lambda_k}, \frac{1}{z - \lambda_\ell} \rangle = \frac{1}{\lambda_k \lambda_\ell - 1}, \]

and then we show that \( J^*_2 = J_{\alpha_2} \). The proof is now complete.

### A.2 Proof of Theorem 4.1

Since \( M(s) \) defined in (4.14) is an inner, we can write (4.8) as

\[ E^*_c = \| M^{-1} Y \tilde{N} - Q \tilde{N} \|_2^2. \]

From Bezout identity (2.6) we have \( M \tilde{X} - Y \tilde{N} = I \), and then \( M^{-1} Y \tilde{N} = \tilde{X} - M^{-1} \). This enables us to write

\[ E^*_c = \| M^{-1} + \tilde{X} - Q \tilde{N} \|_2^2 = \| (1 - M^{-1}) - (1 - \tilde{X} + Q \tilde{N}) \|_2^2. \]

For any \( Q(s) \in \mathbb{R} \mathcal{H}_\infty \) such that \( (1 - \tilde{X} + Q \tilde{N}) \in \mathcal{H}_2 \) then

\[ E^*_c = \| 1 - M^{-1} \|_2^2 + \inf_{Q \in \mathbb{R} \mathcal{H}_\infty} \| 1 - \tilde{X} + Q \tilde{N} \|_2^2. \]

Let denote by \( E_1 \) the first term of above equation and by \( E_2 \) the second term. By noting that

\[ 1 - \frac{s + \tilde{p}_k}{s - \tilde{p}_k} = \frac{-2 \text{Re} \, p_k}{s - \tilde{p}_k}, \]

then
\[ E_1 = \left\| \sum_{k=1}^{n_p} \frac{-2 \text{Re } p_k}{s - p_k} \right\|^2_2 = \sum_{k=1}^{n_p} 4 \text{Re}^2 p_k \left\| \frac{1}{s - p_k} \right\|^2_2. \]

Since
\[ \left\| \frac{1}{s - p_k} \right\|^2_2 = \frac{1}{2 \text{Re } p_k}, \]
we conclude that
\[ E_1 = 2 \sum_{k=1}^{n_p} \text{Re } p_k = 2 \sum_{k=1}^{n_p} p_k. \]

We show that \( E_1 = E_{cm} \). To obtain the closed-form expression of \( E_2 \), we employ the standard partial fraction expansion technique. First, since \( \tilde{N}(z_i) = 0 \) for all \( i \in \mathbb{N}_z \) and \( \tilde{N}(s) \) is left-invertible, we can perform the factorization
\[ \tilde{N}(s) = n(s)a(s), \]
where \( n(s) \) is left-invertible in \( \mathbb{R}H_\infty \) and \( a(s) \) is defined by
\[ a(s) := \prod_{i \in \mathbb{N}_z} \frac{s - z_i}{s + \bar{z}_i}. \]

Since \( a(s) \) is inner, we then obtain
\[ E_2 = \inf_{Q \in \mathbb{R}H_\infty} \left\| \frac{1 - \tilde{X}}{a} + Qn \right\|^2_2. \]

An expansion procedure provides
\[ E_2 = \inf_{Q \in \mathbb{R}H_\infty} \left\| \sum_{i \in \mathbb{N}_z} \frac{s + \bar{z}_i}{s - z_i} \left[ \frac{1 - \tilde{X}(z_i)}{a_i} + R_1 + Qn \right] \right\|^2_2 \]
\[ = \inf_{Q \in \mathbb{R}H_\infty} \left\| \sum_{i \in \mathbb{N}_z} \left[ \frac{s + \bar{z}_i}{s - z_i} - 1 \right] \frac{1 - \tilde{X}(z_i)}{a_i} + R_2 + Qn \right\|^2_2, \]
where
\[ a_i := \prod_{j \in \mathbb{N}_z, j \neq i} \frac{z_j - z_i}{z_j + \bar{z}_i}, \]
\[ R_1(s) := \frac{1 - \tilde{X}(s)}{a(s)} - \sum_{i \in \mathbb{N}_z} \frac{s + z_i}{s - z_i} \frac{1 - \tilde{X}(z_i)}{a_i}, \]
\[ R_2(s) := R_1(s) + \sum_{i \in \mathbb{N}_z} \frac{1 - \tilde{X}(z_i)}{a_i}. \]
Splitting into orthogonal subspaces yields

\[ E_2 = \left\| \sum_{i \in \mathbb{N}_z} \left[ \frac{s + \bar{z}_i}{s - z_i} - 1 \right] \frac{1 - \bar{X}(z_i)}{a_i} \right\|^2_2 + \inf_{Q \in \mathbb{R}H_{\infty}} \| R_2 + Qn \|^2_2. \]

Since \( n(s) \) is left invertible, we may select a \( Q \) such that \( \inf_{Q \in \mathbb{R}H_{\infty}} \| R_2 + Qn \|^2_2 = 0 \). This provides

\[ E_2 = \left\| \frac{2 \Re(z_i)(1 - \bar{X}(z_i))}{a_i} \right\|^2_2. \]

By noting that \( \bar{X}(z_i) = M^{-1}(z_i) \) for all \( i \in \mathbb{N}_z \) we conclude

\[ E_2 = \sum_{i,j \in \mathbb{N}_z} \frac{4 \Re(z_i) \Re(z_j)}{a_ia_j(z_i + z_j)} (1 - M^{-1}(z_i))(1 - M^{-1}(z_j)). \]

We show that \( E_2 = E_{cn} \) by defining \( \alpha_i := 1 - M^{-1}(z_i) \). The proof is now complete.

### A.3 Proof of Theorem 4.2

The proof of the minimum phase part of this theorem mainly follows that of [14, Theorem 3]. Since \( M(s) \) in (4.14) is inner then we may write (4.13) as

\[ E^*_c = \inf_{Q \in \mathbb{R}H_{\infty}} \left\| \begin{bmatrix} W_s(M^{-1} + M^{-1}Y\bar{N} - Q\bar{N}) \\ W_y(X\bar{N} - NQ\bar{N}) \\ M^{-1}Y\bar{N} - Q\bar{N} \end{bmatrix} \right\|^2_2. \]

From Bezout identity (2.6) we obtain identities \( \bar{X} = M^{-1} + M^{-1}Y\bar{N} \) and \( N\bar{X} = X\bar{N} \), such that we may write

\[ E^*_c = \inf_{Q \in \mathbb{R}H_{\infty}} \left\| \begin{bmatrix} W_s(\bar{X} - Q\bar{N}) \\ W_y(N\bar{X} - NQ\bar{N}) \\ \bar{X} - M^{-1} - Q\bar{N} \end{bmatrix} \right\|^2_2 = \inf_{Q \in \mathbb{R}H_{\infty}} \left\| \begin{bmatrix} W\bar{X} - WQ\bar{N} \\ \bar{X} - M^{-1} - Q\bar{N} \end{bmatrix} \right\|^2_2, \]

where

\[ W := \begin{bmatrix} W_s \\ W_y\bar{N} \end{bmatrix}. \]

Based on the orthogonal subspaces splitting we write

\[ E^*_c = E_M + E_Q, \]

where
A.3 Proof of Theorem 4.2

\[ E_M := \left\| \begin{bmatrix} 0 \\ 1 - M^{-1} \end{bmatrix} \right\|_2^2, \]

\[ E_Q := \inf_{Q \in \mathbb{R}^H} \left\| \begin{bmatrix} W \hat{X} - WQ \hat{N} \\ 1 - \hat{X} + Q \hat{N} \end{bmatrix} \right\|_2^2. \]

The proof of Theorem 4.1 shows that

\[ E_M = 2 \sum_{k=1}^{n_p} p_k. \]

Next, we write

\[ E_Q := \inf_{Q \in \mathbb{R}^H} \left\| \begin{bmatrix} W \hat{X} \\ 1 - \hat{X} \end{bmatrix} - \begin{bmatrix} W \\ -1 \end{bmatrix} Q \hat{N} \right\|_2^2 \]

and perform an inner-outer factorization such that

\[ \begin{bmatrix} W(s) \\ -1 \end{bmatrix} = A_1(s)A_o(s), \]

from which we have

\[ |A_o(j\omega)|^2 = 1 + \|W_o(j\omega)\|^2 + \|W_o(j\omega)N(j\omega)\|^2 \]
\[ = 1 + \|W_o(j\omega)\|^2 + \|W_o(j\omega)P(j\omega)\|^2 \]

and \( A_o(\infty) = 1 \). Let define the following norm preserving function

\[ \Gamma(z) := \begin{bmatrix} A_o^S(s) \\ I - A_1(s)A_o^S(s) \end{bmatrix}, \]

i.e., \( \Gamma^\sim(j\omega)\Gamma(j\omega) = I \). By pre-multiplying \( \Gamma \) we obtain

\[ E_Q = \inf_{Q \in \mathbb{R}^H} \left\| \Gamma \begin{bmatrix} W \hat{X} \\ 1 - \hat{X} \end{bmatrix} - \begin{bmatrix} W \\ -1 \end{bmatrix} Q \hat{N} \right\|_2^2 \]
\[ = \inf_{Q \in \mathbb{R}^H} \left\| R_1 - A_oQ \hat{N} \right\|_2^2 + \|R_2\|_2^2, \]

where

\[ R_1 := A_o\hat{X} - A_o^{-H}, \]
\[ R_2 := \begin{bmatrix} W(A_o^H A_o)^{-1} \\ 1 - (A_o^H A_o)^{-1} \end{bmatrix}. \]

Further,

\[ E_Q = \inf_{Q \in \mathbb{R}^H} \left\| A_o\hat{X} - 1 - A_oQ \hat{N} \right\|_2^2 + \|A_o^{-H} - 1\|_2^2 + \|R_2\|_2^2. \]
Let denote
\[ E_{Q_1} := \inf_{Q \in \mathcal{H}_\infty} \left\| A_o \bar{X} - 1 - A_o Q \bar{N} \right\|_2^2, \]
\[ E_{Q_2} := \left\| A_o^{-1} \right\|_2^2 + \left\| R_2 \right\|_2^2. \]

We can similarly follow the partial fraction expansion used in the proof of Theorem 4.1 to show that \( E_{Q_1} = E_{cm} \). Direct calculation by using \( \mathcal{H}_2 \) norm definition yields
\[ E_{Q_2} = -\frac{1}{\pi} \int_{-\infty}^{\infty} (\text{Re} A_o^{-1}(j\omega) - 1) \, d\omega. \]

Since \( A_o^{-1}(\infty) = 1 \) we can apply Lemma 4.3, which gives
\[ E_{Q_2} = -\lim_{s \to \infty} \frac{s[A_o^{-1}(s) - 1]}{s^2 [A_o^{-1}(s)]'} = -\lim_{s \to \infty} s \log A_o^{-1}(s). \]
The last two equalities hold under L’Hopital rule. Application of Lemma 4.4 yields
\[ -\lim_{s \to \infty} s \log A_o^{-1}(s) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \log |A_o^{-1}(j\omega)| \, d\omega. \]

Therefore,
\[ E_{Q_2} = \frac{1}{\pi} \int_{0}^{\infty} \log |A_o(j\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} \log \left( 1 + \|W_s(j\omega)\|^2 + \|W_s(j\omega)P(j\omega)\|^2 \right) \, d\omega. \]
The proof is complete by fact that \( E_M + E_{Q_2} = E_{cm} \).

\section*{A.4 Proof of Theorem 4.3}

Since we set \( M(z) = B(z) \) and \( B(z) \) is an inner function then (4.8) becomes
\[ E_d^* = \inf_{Q \in \mathcal{H}_\infty} \left\| B^{-1} Y \bar{N} - Q \bar{N} \right\|_2^2. \]

From Bezout identity (2.6) we have \( M \bar{X} - Y \bar{N} = I \), or equivalently \( B \bar{X} = Y \bar{N} + I \), and thus \( B^{-1} Y \bar{N} + B^{-1} = \bar{X} \in \mathcal{H}_\infty \). This enables us to write
\[ E_d^* = \inf_{Q \in \mathcal{H}_\infty} \left\| - B^{-1} + \bar{X} - Q \bar{N} \right\|_2^2. \]
For any \( Q \in \mathbb{R} \mathcal{H}_\infty \) such that \( (B^{-1}(\infty) - \tilde{X} + Q\tilde{N}) \in \mathcal{H}_2 \) and together with the fact that \( (B^{-1}(\infty) - B^{-1}) \in \mathcal{H}_2^\perp \), then we may write \( E_3 = E_1 + E_2 \), where

\[
E_1 := \|B^{-1}(\infty) - B^{-1}\|_2^2, \\
E_2 := \inf_{Q \in \mathbb{R} \mathcal{H}_\infty} \|B^{-1}(\infty) - \tilde{X} + Q\tilde{N}\|_2^2.
\]

We shall show that \( E_1 = E_{dn} \) and \( E_2 = E_{dn} \). Since \( B(z) \) is inner then

\[
E_1 = \|B^{-1}(\infty)B(z) - 1\|_2^2 = \left\| \prod_{k=1}^n \frac{\lambda_k z - |\lambda_k|^2}{\lambda_k z - 1} - 1 \right\|_2^2.
\]

Let define

\[
E_{1,N} := \left\| \prod_{k=1}^N \frac{\lambda_k z - |\lambda_k|^2}{\lambda_k z - 1} - 1 \right\|_2^2,
\]

and claim that

\[
E_{1,N} = \prod_{k=1}^N (|\lambda_k|^2 - 1).
\]

We rely on the mathematical induction to prove our claim. For \( N = 1 \), it is true that

\[
E_{1,1} = \left\| \frac{1 - |\lambda_1|^2}{\lambda_1 z - 1} \right\|_2^2 = |\lambda_1|^2 - 1.
\]

Further, let define

\[
\chi(z) := \frac{\bar{\lambda}_N z - 1}{\bar{\lambda}_N z - |\lambda_N|^2}.
\]

It is not difficult to show that \( \chi(z) \) is an inner function. Then by pre-multiplying \( \chi(z) \) to (A.1), we may obtain

\[
E_{1,N} = \left\| \frac{\bar{\lambda}_N - |\lambda_N|^2}{\lambda_N - 1} \left[ \prod_{k=1}^{N-1} \frac{\bar{\lambda}_k z - |\lambda_k|^2}{\bar{\lambda}_k z - 1} - \frac{\bar{\lambda}_N z - 1}{\lambda_N z - |\lambda_N|^2} \right] \right\|_2^2 = |\lambda_N|^2 \left[ E_{1,N-1} + \left\| \frac{\bar{\lambda}_N z - 1}{\lambda_N z - |\lambda_N|^2} - 1 \right\|_2^2 \right].
\]

A direct calculation shows that

\[
\left\| \frac{\bar{\lambda}_N z - 1}{\lambda_N z - |\lambda_N|^2} - 1 \right\|_2^2 = \frac{1}{|\lambda_N|^2} \left\| \frac{|\lambda_N|^2 - 1}{z - \lambda_N} \right\|_2^2 = \frac{|\lambda_N|^2 - 1}{|\lambda_N|^2}.
\]
Hence, we may write $E_{1,N}$ as a recursive expression

$$E_{1,N} = |\lambda_N|^2 E_{1,N-1} + |\lambda_N|^2 - 1.$$  

Suppose it is true that

$$E_{1,N-1} = \prod_{k=1}^{N-1} |\lambda_k|^2 - 1.$$

Then we get

$$E_{1,N} = |\lambda_N|^2 \left[ \prod_{k=1}^{N-1} |\lambda_k|^2 - 1 \right] + |\lambda_N|^2 - 1 = \prod_{k=1}^{N} |\lambda_k|^2 - 1.$$  

And this proves our claim and shows that $E_1 = E_{\Delta m}$. Now we take care of $E_2$, which can be further written as

$$E_2 = \inf_{Q \in \mathbb{R} \mathcal{H}_\infty} \left\| z[B^{-1}(\infty) - \tilde{X}] + B\tilde{N} \right\|_2^2,$$

where $\tilde{N}(z) = z\tilde{N}(z)$. Since $\tilde{N}(\eta_i) = 0$ for all $i \in \mathbb{N}_\eta$ and since $\tilde{N}(z)$ is left-invertible, it can be factorized as $\tilde{N}(z) = n(z)b(z)$, where $n(z)$ is left-invertible in $\mathbb{R} \mathcal{H}_\infty$ and $b(z)$ is defined by

$$b(z) := \prod_{i \in \mathbb{N}_\eta} \frac{z - \eta_i}{\bar{\eta}_i z - 1}.$$

Therefore, we obtain

$$E_2 = \inf_{Q \in \mathbb{R} \mathcal{H}_\infty} \left\| z[B^{-1}(\infty) - \tilde{X}] + Qn \right\|_2^2.$$  

Based on the standard partial fraction expansion procedure, we may write

$$\frac{z[B^{-1}(\infty) - \tilde{X}(z)]}{b(z)} = \sum_{i \in \mathbb{N}_\eta} \frac{1}{v_i} \frac{z[B^{-1}(\infty) - \tilde{X}(\eta_i)]}{b_i} + R_1,$$

where

$$v_i(z) := \frac{z - \eta_i}{\bar{\eta}_i z - 1},$$

$$b_i := \prod_{j \in \mathbb{N}_\eta, j \neq i} \frac{\eta_i - \eta_j}{\bar{\eta}_i \eta_j - 1},$$

and $R_1(z)$ is in $\mathbb{R} \mathcal{H}_\infty$. Then,
\[ E_2 = \inf_{Q \in \mathbb{H}_\infty} \left\| \sum_{i \in \mathbb{N}_\eta} \frac{1}{v_i} [\frac{z}{b_i} - \bar{X}(\eta_i)] + R_1 + Q \right\|^2_2 \]

\[ = \inf_{Q \in \mathbb{H}_\infty} \left\| \sum_{i \in \mathbb{N}_\eta} \left[\frac{1}{v_i} - \bar{\eta}_i\right] \frac{z[B^{-1}(\infty) - \bar{X}(\eta_i)]}{b_i} + R_2 + Qn \right\|^2_2, \]

where

\[ R_2(z) = R_1(z) + \sum_{i \in \mathbb{N}_\eta} \bar{\eta}_i z[B^{-1}(\infty) - \bar{X}(\eta_i)]. \]

Since

\[ \left[\frac{1}{v_i} - \bar{\eta}_i\right] \in \mathcal{H}_2^+, \]

and

\[ (R_2(z) + Q(z)n(z)) \in \mathcal{H}_2, \]

then we have

\[ E_2 = \left\| \sum_{i \in \mathbb{N}_\eta} \left[\frac{1}{v_i} - \bar{\eta}_i\right] \frac{z[B^{-1}(\infty) - \bar{X}(\eta_i)]}{b_i} + \inf_{Q \in \mathbb{H}_\infty} \|R_2 + Qn\|^2_2. \]

Since \( n(z) \) is left invertible we may select a \( Q \) such that

\[ \inf_{Q \in \mathbb{H}_\infty} \|R_2 + Qn\|^2_2 = 0. \]

By fact that

\[ \left[\frac{1}{v_i} - \bar{\eta}_i\right] = \frac{|\eta_i|^2 - 1}{z - \eta_i}, \]

and \( \bar{X} = B^{-1}Y \bar{N} + B^{-1} \) thus \( \bar{X}(\eta_i) = B^{-1}(\eta_i) \) for all \( i \in \mathbb{N}_\eta \), we get

\[ E_2 = \left\| \sum_{i \in \mathbb{N}_\eta} \frac{(|\eta_i|^2 - 1)(B^{-1}(\infty) - B^{-1}(\eta_i))}{b_i(z - \eta_i)} \right\|^2_2. \]

Further, since

\[ \left\| \frac{1}{z - \eta_i} \right\|^2_2 = \frac{1}{|\eta_i|^2 - 1}, \]

we then show that \( E_2 = E_{dn} \) by defining

\[ \beta_i := \prod_{k=1}^{n} \lambda_k - \prod_{k=1}^{n} \bar{\lambda}_k \eta_i - \frac{1}{\eta_i - \bar{\lambda}_k}. \]
A.5 Proof of Theorem 4.4

Since \( M(z) = B(z) \) is an inner function then we may write (4.13) as

\[
E_d^* = \inf_{Q \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} W_s(B^{-1} + B^{-1}Y \tilde{N} - Q \tilde{N}) \\ W_y(X \tilde{N} - NQ \tilde{N}) \\ B^{-1}Y \tilde{N} - Q \tilde{N} \end{bmatrix} \right\|_2^2.
\]

From Bezout identity (2.6) we obtain identities \( \tilde{X} = B^{-1} + B^{-1}Y \tilde{N} \) and \( N \tilde{X} = X \tilde{N} \), such that we may write

\[
E_d^* = \inf_{Q \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} W_s(\tilde{X} - Q \tilde{N}) \\ W_y(N \tilde{X} - NQ \tilde{N}) \\ -B^{-1} + \tilde{X} - Q \tilde{N} \end{bmatrix} \right\|_2^2 = \inf_{Q \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} W \tilde{X} - WQ \tilde{N} \\ -B^{-1} + \tilde{X} - Q \tilde{N} \end{bmatrix} \right\|_2^2,
\]

where

\[
W := \begin{bmatrix} W_s \\ W_y \tilde{N} \end{bmatrix}.
\]

Based on the orthogonal subspaces splitting we write

\[
E_d^* = E_B + E_Q,
\]

where

\[
E_B := \left\| \begin{bmatrix} 0 \\ B^{-1}(\infty) - B^{-1} \end{bmatrix} \right\|_2^2,
\]

\[
E_Q := \inf_{Q \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} W \tilde{X} - WQ \tilde{N} \\ -B^{-1}(\infty) + \tilde{X} + Q \tilde{N} \end{bmatrix} \right\|_2^2.
\]

The proof of Theorem 4.3 shows that

\[
E_B = \prod_{k=1}^{n} |\lambda_k|^2 - 1. \tag{A.2}
\]

Next, we write

\[
E_Q = \inf_{Q \in \mathbb{R}H_\infty} \left\| \begin{bmatrix} W \tilde{X} \\ -1 \end{bmatrix} - \begin{bmatrix} W \tilde{X} - WQ \tilde{N} \\ -B^{-1}(\infty) - \tilde{X} + Q \tilde{N} \end{bmatrix} \right\|_2^2
\]

and perform an inner-outer factorization such that

\[
\begin{bmatrix} W(z) \\ -1 \end{bmatrix} = A_i(z)A_o(z),
\]

where the inner factor \( A_i(z) \) is a stable factor and the outer factor \( A_o(z) \) represents the minimum phase part. Note that \( A_i(z) \) is a column vector transfer function, \( A_o(z) \) is a scalar transfer function, and
\[ |A_0(e^{i\theta})|^2 = 1 + \|W_s(e^{i\theta})\|^2 + \|W_P(e^{i\theta})N(e^{i\theta})\|^2 = 1 + \|W_s(e^{i\theta})\|^2 + \|W_P(e^{i\theta})P(e^{i\theta})\|^2. \]

Note that \(|A_o(\infty)|^2 \neq 1\) since infinity does not lie in the unit circle and we have no properness assumption on \(W_s(z)\) and \(P(z)\). Let define a norm preserving function

\[ \Gamma(z) := \begin{bmatrix} A_o^{-1}(z) \\ I - A_o(z)A_o^{-1}(z) \end{bmatrix}, \]

i.e., \(\Gamma^{-1}(e^{i\theta})\Gamma(e^{i\theta}) = I\). By pre-multiplying \(\Gamma\) we obtain

\[
E_Q = \inf_{Q \in \mathcal{H}_\infty} \left\| \Gamma \left\{ \begin{bmatrix} W \tilde{X} \\ B^{-1}(\infty) - \tilde{X} \end{bmatrix} - \begin{bmatrix} W \\ -1 \end{bmatrix} Q \tilde{N} \right\} \right\|_2^2
= \inf_{Q \in \mathcal{H}_\infty} \|C_1 - A_oQ\tilde{N}\|^2_2 + \|C_2\|^2_2,
\]

where

\[
C_1 := A_o\tilde{X} - A_o^{-H}B^{-1}(\infty),
C_2 := \begin{bmatrix} W(A_o^H A_o)^{-1} \\ 1 - (A_o^H A_o)^{-1} \end{bmatrix} B^{-1}(\infty).
\]

Further, based on the orthonormal subspace splitting we may write

\[ E_Q = E_{Q_1} + E_{Q_2} + E_{Q_3}, \]

where

\[
E_{Q_1} := \inf_{Q \in \mathcal{H}_\infty} \|A_o\tilde{X} - A_o(\infty)B^{-1}(\infty) - A_o\tilde{N}\|^2_2,
E_{Q_2} := \|B^{-1}(\infty)\|^2 \|A_o^{-H} - A_o(\infty)\|^2_2,
E_{Q_3} := \|C_2\|^2_2.
\]

By following the partial fraction expansion as did in the proof of Theorem 4.3 we provide \(E_{Q_1} = E_{dn}\). Next, direct calculation yields

\[
E_{Q_2} = \frac{|A_o(\infty)|^2}{|B(\infty)|^2} + \frac{1}{2\pi|B(\infty)|^2} \int_{-\pi}^{\pi} (|A_o^{-1}(e^{i\theta})|^2 - 2 \text{Re}\{A_o^{-1}(e^{i\theta})A_o(\infty))\}) \, d\theta,
\]

and similarly,

\[
E_{Q_3} = \frac{1}{2\pi|B(\infty)|^2} \int_{-\pi}^{\pi} |A_o^{-1}(e^{i\theta})|^2 \, d\theta.
\]

Therefore,

\[
E_{Q_2} + E_{Q_3} = \frac{|A_o(\infty)|^2}{|B(\infty)|^2} + \frac{2A_o(\infty)}{\pi|B(\infty)|^2} \int_{0}^{\pi} \text{Re} A_o^{-1}(e^{i\theta}) \, d\theta.
\]
Since $\Lambda_0$ is an outer factor, then $\Lambda_0^{-1}$ is in $\mathcal{RH}_\infty$. Invoking Lemma 4.5 yields

$$E_{Q_2} + E_{Q_3} = \frac{|A_0(\infty)|^2 + 1}{|B(\infty)|^2} - \frac{2}{|B(\infty)|^2} = \frac{|A_0(\infty)|^2 - 1}{|B(\infty)|^2}. \quad (A.3)$$

Since

$$|B(\infty)|^2 = \prod_{k=1}^{n_\lambda} \frac{1}{|\lambda_k|^2},$$

then (A.2) together with (A.3) produce

$$E_B + E_{Q_2} + E_{Q_3} = |A_0(\infty)|^2 \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1.$$

We then show that $E_B + E_{Q_2} + E_{Q_3} = E_{4\lambda\infty}$ by application of Poisson-Jensen formula in Lemma 4.1, by fact that $\Lambda_0(z)$ is a stable and minimum phase function, i.e.,

$$|A_0(\infty)|^2 = \exp \left\{ \frac{1}{\pi} \int_0^\pi \log |A_0(e^{j\theta})|^2 \, d\theta \right\}$$

$$= \exp \left\{ \frac{1}{\pi} \int_0^\pi \log \left( 1 + \|W_\lambda(e^{j\theta})\|^2 + \|W_\mu(e^{j\theta}) P(e^{j\theta})\|^2 \right) \, d\theta \right\}. $$

We complete the proof of Theorem 4.4.
References


List of Publications

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- **Journal Papers**
  2. T. Bakhtiar and S. Hara, "$\mathcal{H}_2$ regulation performance limitations for SIMO linear time-invariant feedback control systems." *Automatica*. (provisionally accepted)
  4. S. Hara, T. Bakhtiar, and M. Kanno, "The best achievable $\mathcal{H}_2$ tracking performances for SIMO feedback control systems." Submitted to *Journal of Control Science and Engineering*.

- **International Conference Papers**
  7. T. Bakhtiar and S. Hara, "$\mathcal{H}_2$ regulation performance limits for SIMO feedback control systems." In *Proc. 17th International Symposium on  

\(^1\) The first author won the Young Author Award when presenting this paper.

• Domestic Conference Papers

• Technical Reports
(13) S. Hara and T. Bakhtiar, “$H_2$ tracking performance limitations for SIMO feedback systems: a unified approach to control input penalty case.” Technical Reports. Department of Mathematical Informatics, the University of Tokyo. METR2006-33, May 2006.
Biography

Toni Bakhtiar was born in June 27, 1972 in Tuban, East Java, Indonesia, where he spent most of his childhood. He finished his elementary school in 1985 at SD Negeri Kutorejo 3 Tuban, junior high school in 1988 at SMP Negeri 3 Tuban, and senior high school in 1991 at SMA Negeri 1 Tuban.

In 1991, just after completing his high school, he went to Bogor, West Java, for pursuing the undergraduate study at Department of Mathematics, Bogor Agricultural University, Indonesia. He received a Sarjana Sains degree in mathematics in 1996. Subsequently, he received a Master of Science degree in technical mathematics in 2000 from Department of Control, Optimization, and Stochastic, Delft University of Technology, the Netherlands. Since 2003 he has been working as a Ph.D. student at Department of Information Physics and Computing, the University of Tokyo, Japan. He is the recipient of the Young Author’s Award in the 2004 SICE Annual Conference held in Sapporo, Japan.

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